INMO 2016 Practice Test

Q1 Let $a_1, a_2, \cdots, a_{63}$ be positive real numbers such that $a_1 + a_2 + \cdots + a_{63} = 64$. Then prove that $\sqrt{32a_1 - 1} + \sqrt{32a_2 - 2} + \cdots + \sqrt{32a_{63} - 63} \leq \sqrt{2016}$.

Solution:
Let $x_i = \sqrt{32a_i - i}$ for $i = 1, 2, \cdots, 63$.
Then by RMS-AM inequality, we get:
$LHS = x_1 + x_2 + \cdots + x_{63} \leq \sqrt{63 \cdot (x_1^2 + x_2^2 + \cdots + x_{63}^2)}$
$= \sqrt{63 \cdot (32 \cdot [a_1 + a_2 + \cdots + a_{63}] - [1 + 2 + \cdots + 63])} = \sqrt{2016}$.

Q2 Determine all n such that $n = (\tau(n))^2$, where $\tau(n)$ denotes the number of positive divisors of n.

Solution: $n = 1$ is a trivial solution, so let us assume $n > 1$.
Let $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$ be the prime factorization of n.
$\therefore p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} = n = \tau(n)^2 = (k_1 + 1)^2 (k_2 + 1)^2 \cdots (k_t + 1)^2$
Since n is a perfect square, all the k_i's are even; so $\tau(n)$ is odd.
Hence, n is also odd; so all p_i's are ≥ 3.
Lemma: $p_i^{k_i} \geq (k_i + 1)^2$ for any $p_i \geq 3, k_i \geq 2$.
Proof: We use induction on k_i.
For $k_i = 2$, $p_i^{k_i} = p_i^2 \geq 3^2 = (k_i + 1)^2$, proving the induction base.
For the induction step, we note that
$\frac{p_i^{k_i+1}}{p_i^{k_i}} = p_i > 2 \geq \frac{(k_i+2)^2}{(k_i+1)^2}$
So if $p_i^{k_i} \geq (k_i + 1)^2$, then $p_i^{k_i+1} \geq (k_i + 2)^2$; as required.
In addition, note that the induction base is a strict inequality for any $p_i > 3$; while the induction step is always strict inequality. Thus, the lemma is an equality only for $p_i = 3, k_i = 2$.

Applying the lemma for each $i = 1, 2, \cdots, t$, and combining, we get
$p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t} \geq (k_1 + 1)^2 (k_2 + 1)^2 \cdots (k_t + 1)^2$
with equality only if the prime factorization contains just one term: $p_1 = 3, k_1 = 2$, implying $n = 9$.
Thus, the solution set is $\{1, 9\}$.

Q3 In $\triangle ABC$, H is the orthocenter, and K is the foot of the perpendicular from H on the internal bisector of $\angle A$. Prove that K is collinear with the midpoints of AH and BC.

Solution:
Let X, D be the midpoints of segments AH, BC respectively, and O be the circumcenter of ΔABC.
Since ΔAPK is isosceles; we get $\angle XKA = \angle XAK$.
Also, H and O are isogonal conjugates; hence $\angle XAK = \angle KAO$.
$\therefore \angle XKA = \angle KAO$, implying $XK||AO$.
However, $AODX$ is a parallelogram, therefore $AO||XD$.
Hence $XK||XD$, implying $X - K - D$; QED.

Q4 Let S be a set of n distinct points in a plane; and T be a set of $n+1$ distinct triangles, all of whose vertices belong to S. Prove that there exist 2 triangles in T that have exactly 1 vertex in common.

Solution:
We will use strong induction to prove a stronger statement:
Claim: Let S be a set of n distinct points in a plane; and T be a set of at least $n + 1$ triangles, all of whose vertices belong to S. Then either the elements of T are not all distinct, or there exist 2 triangles in T such that they have exactly 1 vertex in common.

Proof: For the induction base, note that for $n = 1, 2, 3, 4$, we have $\binom{n}{3} < n + 1$; so there cannot be $n + 1$ distinct triangles in T.
For the induction step, let the claim be true for all numbers up to $n - 1$; we wish to prove it for n.
If possible, let there be a set T containing at least $n + 1$ distinct triangles, none of which share exactly 1 vertex. In other words, if two triangles in T share a vertex, then they share exactly 2 vertices. Together, these triangles use at least $3n + 3$ vertices from the set S containing only n vertices. So by pigeonhole principle, there exists a vertex $A \in S$ which belongs to at least 4 different triangles.
Let us call vertices X, Y 'adjacent' if there is a triangle in T that has both of them as vertices. Let $\{B_1, B_2, \cdots, B_k\}$ be the set of all vertices adjacent to A. We note that $k \geq 4$, else we wouldn't be able to form 4 distinct triangles containing A.
WLOG let AB_1B_2 be one of the triangles containing A. Consider a second triangle containing A; one of its vertices must be B_1 or B_2, WLOG let it be B_1; and let the third vertex be called B_3.
So we have triangles AB_1B_2 and AB_1B_3. There are at least 2 more triangles containing A. At most one of them could be AB_2B_3; but

that still leaves at least yet another triangle with a vertex, say B_4. Since this triangle has to share exactly 2 vertices with AB_1B_2 and AB_1B_3, this triangle must have B_1 as one of its vertices.
So now, we have triangles AB_1B_2, AB_1B_3 and AB_1B_4. (Note that this now eliminates the possibility of AB_2B_3 being a triangle in T, else it would share exactly 1 vertex with AB_1B_4.)
Similarly, for $i = 2, 3, \cdots, k$, B_i can be a vertex of only AB_1B_i. As a result, the $(k+1)$ vertices in the set $V = \{A, B_1, B_2, \cdots, B_k\}$ are used in exactly the $(k-1)$ triangles $\{AB_1B_2, AB_1B_3, \cdots, AB_1B_k\}$, and none of them are adjacent to any of the $n-(k+1)$ vertices in $S \backslash V$, which form at least $(n+1)-(k-1) = n-k+2$ triangles. By the induction hypothesis, $S \backslash V$ satisfies the claim; therefore, so does the set S. Thus our proof is complete.

Q5 In the parallelogram $ABCD$, points E and F are on sides AB and CD respectively. AF intersects ED in G. EC intersects FB in H. GH produced intersects AD in L and BC in M. Prove that $DL = BM$.

Solution: Firstly, we note that it suffices to prove that GH passes through the point of intersection of the diagonals of $ABCD$. So we need to prove that GH is collinear with the midpoint O of AC.
Let $DE \cap FB = P, EC \cap AF = Q$.
Let $AE = x, EB = y, BF = z, FD = w$. Then $x + y = z + w$.
Applying Menelaus' theorem in ΔEPA for transversal $F - H - B$, we get $\frac{EH}{HP} \cdot \frac{PF}{FA} \cdot \frac{AB}{BE} = -1$
But $\frac{PF}{FA} = \frac{PC}{CE}$
$\therefore \frac{EH}{HP} \cdot \frac{PC}{CE} \cdot \frac{AB}{BE} = -1$
$\therefore \frac{PC}{HP} = -\frac{CE}{EH} \cdot \frac{EB}{BA} = -\frac{y+z}{y} \cdot \frac{y}{x+y}$
$\therefore \frac{PC}{HP} = -\frac{y+z}{x+y}$
By symmetry, we can do similar calculations for ΔFQB with transversal $E - H - C$, to get
$\frac{QB}{HQ} = -\frac{z+y}{w+z}$
$\therefore \frac{PC}{HP} = \frac{QB}{HQ}$, implying $PQ || BC || AD$.
So by B.P.T. we get $\frac{CH}{HP} = -\frac{BC}{PQ} = -\frac{AD}{PQ} = -\frac{AG}{GP}$
$\therefore \frac{AO}{OC} \cdot \frac{CH}{HP} \cdot \frac{PG}{GA} = -1$
since $AO = OC$.
So by converse of Menelaus' Theorem, we get $G - O - H$ as required.

Q6 Find all polynomials $f(x)$ with real coefficients, such that for all $x \in \mathbb{R},\ f(x^2) + f(x)f(x+1) = 0$.

Solution:
Let $g(x) = f(x+1)$ and $h(x) = f(x^2)$. Then $h(x) = -f(x)g(x)$.
Let the degree of f be n. So f has n roots in $\mathbb{C}$, possibly repeated.
Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the complex roots of f.
Then $\alpha_1 - 1, \alpha_2 - 1, \cdots, \alpha_n - 1$ are the roots of g.
Hence, the roots of h are $\alpha_1, \alpha_2, \cdots, \alpha_n, \alpha_1 - 1, \alpha_2 - 1, \cdots, \alpha_n - 1$.
Now, note that $f(\alpha_i) = 0 \Rightarrow f(\alpha_i^2) = h(\alpha_i) = -f(\alpha_i)g(\alpha_i) = 0$.
So if α_i is a root of f, so is α_i^2.
If $0 < |\alpha_i| < 1$ or $1 < |\alpha_i|$, then $\alpha_i, \alpha_i^2, \alpha_i^4, \cdots$is an infinite non-repeating sequence of roots, implying f is the zero polynomial.
To find other solutions, let $S = \{z : z \in \mathbb{C}; |z| \in \{0, 1\}\}$, and let $\alpha_i \in S$ for $i = 1, 2, \cdots, n$.
We note that since $\beta = \alpha_i - 1$ is a root of h, then $0 = h(\beta) = f(\beta^2)$, implying β^2 is one of the α_i's.
So $\beta^2 \in S \Rightarrow |\beta^2| \in \{0, 1\} \Rightarrow |\beta| \in \{0, 1\} \Rightarrow \beta \in S$.
Hence, $\alpha_i - 1 \in S$ for $i = 1, 2, \cdots, n$.
Thus all roots α_i of f belong to the set:
$T = S \cap \{z + 1 : z \in S\} = \{0, 1, cis(\frac{\pi}{3}), -cis(\frac{\pi}{3})\}$.
However, if $\alpha_i = \pm cis(\frac{\pi}{3})$, then $\alpha_i^2 = \pm cis(\frac{2\pi}{3}) \notin T$, contradicting the previously shown fact that α_i^2 is also a root of f.
So the only possible roots of f are 0 and 1.
Also, comparing the leading coefficients in the equation $f(x^2) + f(x)f(x+1) = 0$, we see that the same has to be -1.
$\therefore f(x) = -x^k(x-1)^l$ for some non-negative integers k, l.
$\therefore g(x) = -(x+1)^k x^l$
$\therefore h(x) = -(x+1)^k x^{k+l}(x-1)^l$
$= f(x^2) = -x^{2k}(x^2-1)^l = -(x+1)^l x^{2k}(x-1)^l$.
Comparing the exponents, we see that $k = l$ is forced.
Thus the possible solutions are:
$f(x) \equiv 0$; and
$f(x) = -x^k(x-1)^k$ for any non-negative integer k.