INMO 2016 Practice Test II

Q1 Let $S = \{1, 2, \cdots, n\}$. A subset G of S is called a good set if and only if $|G| \in G$. A subset E of S is said to be an excellent set if and only if E is good and has no proper good subset. Find the number of excellent subsets of S.

Solution: For $k = 1, 2, \cdots, n$, let us count the number of excellent subsets of size k. If E is such a set, then E is good; so $k \in E$.
Now, for any natural number $l < k$, if $l \in E$, then we can consider a proper subset of E that contains exactly l elements, including l. Such a subset would be good; which contradicts E being excellent. Hence, all the elements of E belong to $\{k, k+1, \cdots, n\}$. We also see that this criterion is sufficient for E to be an excellent set.
So, the number of $k-$element excellent sets is given by $\binom{n-k}{k-1}$; since we have to choose the remaining $k-1$ elements other than k itself. Combining the from $k = 1, 2, \cdots, n$, we see that the total number of excellent subsets for S is given by:
$E_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-k}{k-1} + \cdots +$
Note that the terms of the above series will be zero, for $k > \frac{n}{2}$.
We observe that $E_n + E_{n+1} = E_{n+2}$, since by Pascal's identity:
$\binom{n-k}{k-1} + \binom{(n+1)-(k+1)}{(k+1)-1} = \binom{n-k}{k-1} + \binom{n-k}{k} = \binom{n+1-k}{k}$
Further, $E_1 = 1; E_2 = 1$. Hence we see that E_n is essentially the Fibonacci sequence: $E_n = F_{n-1}$.

Q2 Let C be a point on a semicircle of diameter AB and let D be the mid point of arc AC. Denote by E the projection of the point D on the line BC. Let F be the intersection of the line AE with the semicircle. Prove that line BF bisects the line segment DE.

Solution: Let BF intersect DE at point X.
Since $DE \perp BC$ and $AC \perp BC$, we have $DE || AC$.
$\therefore \angle EDC = \angle DCA = \angle DAC$; so by the converse of tangent-secant theorem, ED is a tangent to the given circle at D.
$\therefore XD^2 = XF \cdot XB$.
Consider the circle with diameter BE; it passes through F.
Also, $\angle DEA = \angle EAC = \angle FAC = \angle FBC = \angle FBE$; so by the converse of tangent-secant theorem, DE is a tangent to this circle at point E.
$\therefore XE^2 = XF \cdot XB$.
Hence $XD = XE$; implying X is the midpoint of DE as required.

Q3 For any $m, n \in \mathbb{N}$, let (m, n) denote their GCD.
For any $n \in \mathbb{N}$ let us define $\psi(n) = \sum_{i=1}^{n} (n, i)$. Then prove that $\psi(mn) = \psi(m) \cdot \psi(n)$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$.

Solution: We use the following lemma:
Lemma: If $(m, n) = 1$, then $\forall i \in \mathbb{N} : (mn, i) = (m, i) \cdot (n, i)$.
Proof: Let p be any prime; and α, β, γ be the exponents of p in the prime factorizations of m, n and i respectively.
Hence, the exponent of p in the factorization of (mn, i) will be $\min(\alpha + \beta, \gamma)$. Similarly, the exponents for $(m, i), (n, i)$ will be $\min(\alpha, \gamma), \min(\beta, \gamma)$ respectively.
Since $(m, n) = 1$, one of α, β must be zero; WLOG let $\beta = 0$.
$\therefore \min(\alpha + \beta, \gamma) = \min(\alpha, \gamma) = \min(\alpha, \gamma) + \min(\beta, \gamma)$
since $\min(\beta, \gamma) = 0$. Thus the prime factorizations of (mn, i) and $(m, i) \cdot (n, i)$ are identical; implying they are equal; QED.

Now, returning to the given problem:
$\psi(mn) = \sum_{i=1}^{mn} (mn, i) = \sum_{i=0}^{mn-1} (m, i) \cdot (n, i)$
(Note that we have shifted the range of the summation index, but since $(mn, 0) = (mn, mn)$; the value of the sum is unchanged.)
Since $(m, qm + r) = (m, r)$, we can regroup the terms of the summation, based on their remainders modulo m, to get:
$\psi(mn) = \sum_{i=0}^{mn-1} (m, i) \cdot (n, i)$
$= \sum_{r=0}^{m-1} \sum_{q=0}^{n-1} (m, qm + r) \cdot (n, qm + r)$
$= \sum_{r=0}^{m-1} \left[(m, r) \sum_{q=0}^{n-1} (n, qm + r) \right]$
The set of values $(qm + r)$ for $q = 0, 1, \cdots, (n-1)$ forms a complete set of residues modulo n, since $(m, n) = 1$.
$\therefore \sum_{q=0}^{n-1} (n, qm + r) = \sum_{t=0}^{n-1} (n, t) = \psi(n)$.
$\therefore \psi(mn) = \sum_{r=0}^{m-1} [(m, r)\psi(n)] = \psi(n) \sum_{r=0}^{m-1} (m, r) = \psi(n) \cdot \psi(m)$, as required.

Q4 Find all functions $f : \mathbb{Z} \to \mathbb{R}$ such that
$f(m+n) + f(m-n) = \frac{1}{2}(f(2m) + f(2n))$ for all $m, n \in \mathbb{Z}$

Solution: Putting $m = n = 0$, we get $f(0) = 0$.
Now, putting $n = 0$ we get $2f(m) = \frac{1}{2}f(2m)$.
Thus the given equation can be rewritten as:
$f(m+n) + f(m-n) = 2(f(m) + f(n))$ for all $m, n \in \mathbb{Z}$
Let $f(1) = k$. Then putting $n = 1$, we get:
$f(m+1) + f(m-1) = 2[f(m) + k]$.
We notice that this is a recurrence relation for f; with the initial conditions $f(0) = 0, f(1) = k$.
Also, we notice that $f(m) = km^2$ satisfies the same, so it must be the only solution to the above equation and initial conditions. (Alternately, the same can be established by induction as well.)
Therefore the only solution is: $f(m) = km^2$ for some $k \in \mathbb{R}$.

Q5 Let $x, y, z \in \mathbf{R}, x, y, z \geq 0, xyz = 1$. Then show that
$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+x)(1+z)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$.

Solution: Cross-multiplying to clear the denominators, and using $xyz = 1$, we get the equivalent statement:
$4\sum x^3(1+x) \geq 3(1+x)(1+y)(1+z) = 3(2 + \sum x + \sum xy)$
$\Leftrightarrow 4\sum x^4 + 4\sum x^3 \geq 3\sum x + 3\sum xy + 6$.
We wish to decompose the above into several smaller inequalities. For that, we regroup the terms, and multiply by a suitable power of (xyz) wherever necessary, to get the equivalent:
$(3\sum x^4) + (\sum x^4) + (2\sum x^3) + (2\sum x^3)$
$\geq (3\sum x^2yz) + (\sum x^{\frac{5}{3}}y^{\frac{5}{3}}z^{\frac{2}{3}}) + (2\sum x^{\frac{4}{3}}y^{\frac{4}{3}}z^{\frac{1}{3}}) + (6xyz)$
The first three corresponding brackets on the LHS and RHS are related by a direct application of the rearrangement inequality, and the last bracket, by A.M.-G.M. inequality. Hence proved.

Q6 Let D be an arbitrary point of segment AB of given $\triangle ABC$. Let E be the interior point where CD intersects external common tangent (other than AB) to the incircles of triangles ADC and BDC. Find the locus of E as D moves on segment AB.

Solution: Let Γ_1, Γ_2 be the incircles of ΔADC, ΔBDC respectively. Let K, L be the points of tangency of Γ_1 and Γ_2 respectively, on line DC. Let the external common tangent of Γ_1, Γ_2 other than AB, meet AB at point X.
Then we see that Γ_1, Γ_2 are the incircle and an excircle for ΔXDE, in some order; implying that KL has the same midpoint as DE.
In other words, $EK = DL$; i.e. $DE = DK + KE = DK + DL$.
Now, by the known incircle configuration in $\Delta ADC, \Delta BDC$,
$DK = \frac{AD+DC-AC}{2}$, and $DL = \frac{BD+DC-BC}{2}$.
$\therefore DE = DC + \frac{AD+BD-AC-BC}{2} = DC + \frac{AB-AC-BC}{2}$.
$\therefore EC = DC - DE = \frac{AC+BC-AB}{2}$, a constant independent of D.
Hence, the locus of E is a circle centered at C, of radius $\frac{AC+BC-AB}{2}$.