INMO 2017: Practice Test 1

Time : 4 hours **December 02, 2016**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Draw neat Geometry diagrams.
- Each question is worth 17 marks. Total marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.
- Mathematical reasoning will be taken into consideration while assessing the answers.

1. $ABCD$ is a square of side length 17 units. Points $P_1, P_2, \cdots, P_{16}$ are taken on side AB in that order, dividing the side into 17 segments of equal length. Similarly, points Q_1 to Q_{16} are taken on side BC, points R_1 to R_{16} are taken on side CD, and points S_1 to S_{16} are taken on side DA. Find all quadrilaterals of the form $P_iQ_jR_kS_l$ whose perimeter is an integer. (Each of the indices i, j, k, l is chosen from $\{1, 2, \cdots, 16\}$ with repetitions allowed.)

2. Let AB be a straight road of length n, where $n \in \mathbb{N}$. At regular intervals of length 1 between the endpoints A and B, there is a small puddle; excluding the endpoints themselves (so a total of $n - 1$ puddles). A man starts from A and travels towards B, always taking steps of length either $\frac{1}{4}$ or $\frac{3}{4}$. In how many ways can he complete his journey while avoiding all the puddles?

3. Find all integers k such that **exactly two** roots of the following polynomial are integers:

$$f(x) = 2x^3 + (2k - 5)x^2 - (k + 10)x + (9 - 3k)$$

4. Construct ΔABC using a straightedge and compass, given the lengths of side AB, the altitude h_a from point A, and the symmedian m'_b from point B.

5. Let x, y, z be non-negative real numbers such that $xyz = 1$. Prove that:

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}$$

6. (20 marks) ABC is an equilateral triangle with side length 6 units. As shown in the figure, points $P_1, P_2, \cdots, P_5$ are taken on side BC in that order; dividing the side into 6 segments of equal length. Similarly, points Q_1 to Q_5 are taken on side CA, and points R_1 to R_5 are taken on side AB. Count the number of triangles of the form $P_iQ_jR_k$ such that their centroid coincides with the centroid of ΔABC. (Each of the indices i, j, k is chosen from $\{1, 2, \cdots, 5\}$ with repetitions allowed.)