INMO 2017: Practice Test 3

Time : 4 hours **December 25, 2016**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Draw neat Geometry diagrams.
- Each question is worth 17 marks. Total marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.
- Mathematical reasoning will be taken into consideration while assessing the answers.

Questions:

1. Let $k \in \mathbb{N}$. Prove that there exist integers x, y neither of which is divisible by 3, such that $x^2 + 2y^2 = 3^k$.

2. Let $ABCD$ be a quadrilateral inscribed in circle k. Lines AB and CD intersect at point E such that $AB = BE$. Let F be the intersection point of tangents on circle k in points B and D, respectively. If the lines AB and DF are parallel, prove that A, C and F are collinear.

3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy:

$$x^2(f(x) + f(y)) = (x + y)(f(yf(x)))$$

for all $x, y \in \mathbb{R}^+$.

4. Let $n \in \mathbb{N}$. In how many ways can the squares of a $n \times n$ board be coloured in black and white, so that any region of 2×2 adjacent squares on the board will have exactly two white and two black squares?

5. In acute-angled ΔABC, let E, F be the feet of the altitudes from B to AC and C to AB respectively. Let the reflection of F in the lines AC and BC be points S and T respectively. Let the circumcircle of ΔCST intersect line AC a second time in point X, and let the circumcenter of ΔCST be K. Prove that $XK \perp EF$.

6. In the XY coordinate plane, a needle of unit length is originally lying along the X-axis, with its blunt end at $(0, 0)$, and sharp end at $(1, 0)$. You are allowed to do the following operation: Keeping any one end of the needle fixed, rotate the needle around that end by exactly 45 degrees (either clockwise or counter-clockwise). Is it possible to repeatedly apply this operation, and reverse its original position (or in other words, its blunt end should be at $(1, 0)$, and sharp end should be at $(0, 0)$)?