INMO 2017: Practice Test 4

Time : 4 hours **January 8, 2017**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Draw neat Geometry diagrams.
- Each question is worth 17 marks. Total marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.
- Mathematical reasoning will be taken into consideration while assessing the answers.

Questions:

1. Let $m, n \in \mathbb{Z}$. If $5(m^4 + n^4) + 12mn(m^2 + n^2) + 30m^2n^2$ is prime, then find m, n.

2. In ΔABC, a point M is taken on side BC such that the centroid of ΔABM is concyclic with A, C, M, and the centroid of ΔACM is concyclic with A, B, M. Prove that M is equidistant from the midpoints of the sides AB and AC.

3. Consider an $n \times n$ board whose squares are colored in black and white; such that exactly three of the corner squares are white, and the fourth one is black. Prove that there exists a 2×2 square which contains an odd number of white squares.

4. Let $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 + c^2 + d^2 = 1$. Find the maximum possible value of the expression $\sum_{sym} (a + b)^4$, where the sum is symmetric, i.e. there are $\binom{4}{2} = 6$ terms in all.

5. Let N be a point on the longest side AC of a triangle ABC. The perpendicular bisectors of AN and NC intersect AB and BC respectively in K and M. Prove that the circumcenter O of $\triangle ABC$ lies on the circumcircle of triangle KBM.

6. Let $n \in \mathbb{N}$. Find the largest possible length of a sequence $a_1, a_2, \cdots, a_k$ that can be formed using numbers from $\{1, 2, \cdots, n\}$ with repetitions allowed, such that:

   1. Consecutive terms are unequal; i.e. $a_i \neq a_{i+1}$ for any i
   2. No four terms form an alternating pattern of two values; i.e. there do not exist $i < j < k < l$ such that $a_i = a_k$ and $a_j = a_l$.

   Your answer will be in terms of n.