INMO Practice Test

Time : 4 hours **December 29, 2019**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions. Draw neat Geometry diagrams.
- Each question is worth 17 marks. Total marks: 102.
- Answer to each question should start on a new page. Clearly indicate the question number.
- Mathematical reasoning will be taken into consideration while assessing the answers.

Questions:

1. Prove that for any $m \in \mathbb{N}$, the number of integer solutions to the equation $x_1^2 + x_2^2 + \cdots + x_{2020}^2 = m$ is divisible by 8.

2. Let ΔABC be acute-angled, and D be a point on seg BC. Let I_1, I_2 be the incenters of $\Delta ABD, \Delta ADC$ resp.; and let O_1, O_2 be the circumcenters of $\Delta AI_1D, \Delta AI_2D$ resp. If I_1O_2 and I_2O_1 meet at point P, prove that $PD \perp BC$.

3. Let $n \in \mathbb{N}$. If S is a set of integers such that:

    1. For each $a \in S$, $|a| \leq n$; and
    2. For any $a, b, c \in S$ (not necessarily distinct), $a + b + c \neq 0$.

    What is the maximum possible size of S?

4. Let n be an odd number greater than 3, and let $a_1, a_2, \cdots, a_n$ be positive integers such that $GCD(a_1, a_2, \cdots, a_n) = 1$. If $P = a_1 \times a_2 \times a_3 \times \cdots \times a_n$, then find all possible values of $GCD(a_1^n + P, a_2^n + P, \cdots, a_n^n + P)$.

5. Given a fixed circle Γ and a fixed chord PQ, let X be a variable point on seg PQ. Let A be the midpoint of minor arc PQ; let AX meet Γ a second time at S, and let the perpendicular line to AX at X meet Γ at point T such that P, T are on opposite sides of line AX. Find the locus of the midpoint of chord ST as X varies on seg PQ.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + 2f(x)f(y)) = f(x) + 2xf(y)$ for all $x, y \in \mathbb{R}$.