INMO 2022 Practice Test

Q1 Let $S = \{1, 2, \cdots, n\}$. A subset G of S is called a good set if and only if $|G| \in G$. A subset E of S is said to be an excellent set if and only if E is good and has no proper good subset. Find the number of excellent subsets of S.

Q2 Let C be a point on a semicircle of diameter AB and let D be the mid point of arc AC. Denote by E the projection of the point D on the line BC. Let F be the intersection of the line AE with the semicircle. Prove that line BF bisects the line segment DE.

Q3 For any $m, n \in \mathbb{N}$, let (m, n) denote their GCD.
For any $n \in \mathbb{N}$ let us define $\psi(n) = \sum_{i=1}^{n} (n, i)$. Then prove that $\psi(mn) = \psi(m) \cdot \psi(n)$ for all $m, n \in \mathbb{N}$ such that $(m, n) = 1$.

Q4 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{Q}$:

$$\begin{aligned} 4f(x+y+z) = 3[f(x+y) + f(y+z) + f(z+x)] \\ -[f(x-y) + f(y-z) + f(z-x)] \end{aligned}$$

Q5 Let $x, y, z \in \mathbf{R}, x, y, z \geq 0, xyz = 1$. Then show that

$$\frac{1}{x^4(1+y)(1+z)} + \frac{1}{y^4(1+x)(1+z)} + \frac{1}{z^4(1+x)(1+y)} \geq \frac{3}{4}$$

Q6 Let D be an arbitrary point of segment AB of given $\triangle ABC$. Let E be the interior point where CD intersects external common tangent (other than AB) to the incircles of triangles ADC and BDC. Find the locus of E as D moves on segment AB.