MPI RMO Selection test 2020

Each question has an integer answer from 000 to 999.
Maximum marks 50; total time 90 minutes.

Questions

Section A: 2 marks per question

1. ABC is an isosceles triangle with base $BC = 16$ and $AB = AC = 10$. Let P be a variable point inside the triangle, and x, y, z be the distances from P to the sides BC, CA, AB respectively. Find the maximum possible value of $xy + yz + zx$.

2. An ice-cream vendor has 3 different flavours. A man wants to buy 5 double-scoop ice-cream cones; such that each cone has a different flavour on its top scoop than its bottom scoop. How many possible orders can he give to the vendor?

3. Let S be a set of 16 points arranged in a regular 4×4 square grid formation. How many non-degenerate triangles can be formed whose vertices lie in S? (Non-degenerate meaning, the vertices are not collinear.)

4. A natural number is called unique if its decimal representation is of the form $(d_1 d_2 \cdots d_k d_1 d_2 \cdots d_k)$ with $d_1 \neq 0$. In other words; the same sequence of digits is repeated twice. For eg, 2020 is unique. If the sum of the first 50 unique numbers is N, find the remainder of N on dividing by 1000.

5. Circles O_1 and O_2 of radii 8 and 18 respectively, intersect at point K such that $\angle O_1 K O_2 = 60°$. If a common tangent line to the two circles touches O_1, O_2 at points A, B respectively, then find the length of seg AB.

Section B: 3 marks per question

6. Given ΔABC of area 60; let P, Q, R be points on the sides BC, CA, AB respectively, such that $\frac{BP}{PC} = \frac{2}{3}$; $\frac{CQ}{QA} = \frac{1}{3}$ and $\frac{AR}{RB} = \frac{1}{2}$. Find the area of ΔPQR.

7. Given that the polynomials $P(x) = x^3 - 3x - 2$ and $Q(x) = x^3 - 5x^2 + 8x - 4$ have a common root; find the sum of the largest root of Q and the smallest root of P.

8. For any natural number n, let $z(n)$ denote the number of zeros at the end of the decimal representation of n. For eg, $z(5!) = 1$. Find the value of $\sum_{i=15}^{85} z\left(\binom{i}{10}\right)$.

9. For any natural number n, let $d(n)$ denote the sum of the digits of n. Count the number of ordered pairs of 3-digit natural numbers (n_1, n_2) such that $n_1 < n_2$ and $n_1 + d(n_1) = n_2 + d(n_2)$.

10. Let k be an integer. Consider the polynomials $P(x) = -x^2 + 10x - 23$, $Q(x) = x + k$ and $R(x) = x^2 + 6x + 11$. Find the number of values of k such that $P(x) < Q(x) < R(x)$ for all real numbers x.

Section C: 5 marks per question

11. Let S be a set of 11 points in a plane, such that no three of them are collinear. What is the maximum number of segments that can be drawn, having endpoints in S, such that no three segments form a triangle whose vertices are in S?

12. Find the largest possible G.C.D of $(n^2 + 2)$ and $(n^3 + 3)$, where n is any integer.

13. x, y, z are real numbers such that $4x + 6y + 9z = 44$ and $4x^2 + 4y^2 + 9z^2 = 88$. Find the value of $x + y + z$.

14. A monk is a special chess piece which can move in a straight line by either 1 square or 2 squares in horizontal or vertical direction. Given a 6×10 board, what is the maximum number of monks that can be placed such that no two monks attack each other?

15. Points A, B, C lie on a line l such that $A - B - C$. A circle Γ passes through A and B, and the tangent to Γ from point C touches the circle at point T. Given that TB is the internal angle bisector of $\angle ATC$, and $TB = 27$, $TC = 18$, find the length of TA.

Hints and Answers

1. **[024]** The optimum position of P would be on the altitude from vertex A. Write the desired expression as a function of only x; which turns out to be a quadratic expression.

2. **[252]** There are 6 valid flavour combinations for the double-scoop icecream. This essentially becomes the standard problem of choosing 5 objects from 6 types with repetitions allowed.

3. **[516]** First count all the possible triples of points; then eliminate the collinear cases.

4. **[725]** Split and compute the sum as separate A.P.s of 1-digit and 2-digit unique numbers.

5. **[012]** First use cosine rule to get the length of O_1O_2. Then consider the altitude from O_1 to seg O_2B to complete the calculations.

6. **[020]** Find the areas of the three surrounding triangles in terms of the ratios of the side arms, using the formula $\frac{1}{2}bc \cdot \sin A$ when the angle is common.

7. **[001]** Note that the common root is also a root of $P(x) - Q(x) = 0$ which is a quadratic equation. Find the common root first, then use it to find all other roots.

8. **[030]** The number of zeros in $\binom{i}{10}$ is determined by the number of extra occurrences of 5 in the prime factorization of the numerator, compared to the denominator. Note that only when a power of 5 such as 25, 50, 75 is present in the numerator, will this difference be 1; else it is always 0.

9. **[072]** If the two numbers are $(a_1b_1c_1)$ and $(a_2b_2c_2)$, we get $101(a_2 - a_1) = 11(b_1 - b_2) + 2(c_1 - c_2)$ which forces $a_2 - a_1 = 1$; $b_1 = 9, b_2 = 0$ and $c_1 - c_2 = 1$.

10. **[007]** The discriminants of $R(x) - Q(x)$ and $Q(x) - P(x)$ should be positive; which leads to limits on the allowed values of k.

11. **[030]** Dividing the 11 points in two sets of 5 and 6 points respectively, we can join all points from the first set with all points from the second set, without forming a triangle. One can show by induction that joining 31 edges will always form a triangle.

12. **[017]** Using repeated application of Euclid's Algorithm, any common divisor is forced to be also a divisor of 17. This value is achieved when $n = 10$.

13. **[007]** Writing the given expression as $2 \cdot 2x + 3 \cdot 2y + 3 \cdot 3z$, we get the equality case of Cauchy Schwartz inequality. This forces $2x : 2y : 2z = 2 : 3 : 3$.

14. **[020]** Assuming each row to be of length 6, it can fit exactly 2 monks. Further by staggering the placements on each row, we can avoid the monks from attacking each other along any column also.

15. **[054]** We see that $AB = TB = 27$, and $CA \times CB = CT^2$; from which we get CA, CB.