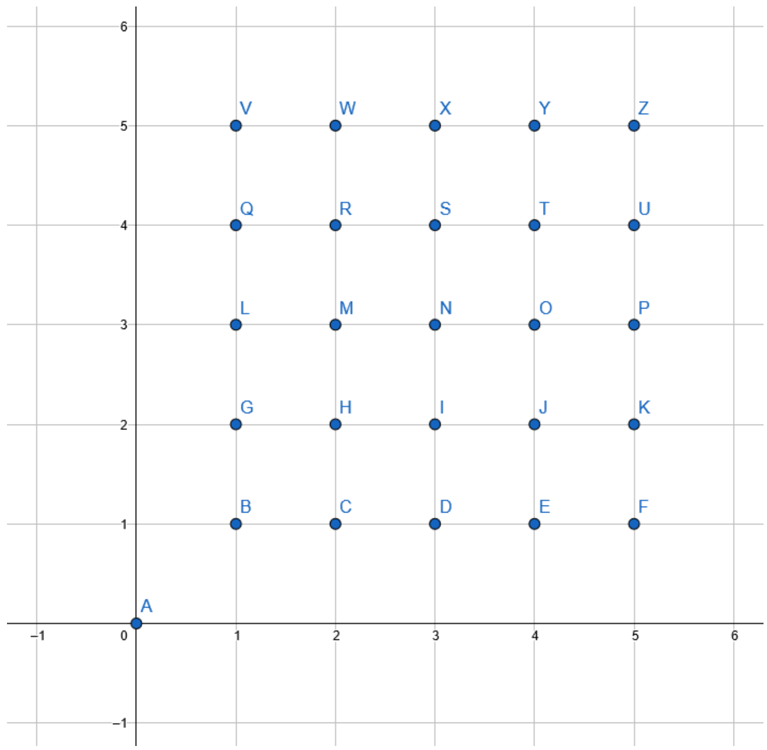
MPI RMO Selection test 2021

Each question has an integer answer from 000 to 999.
Maximum marks 50; total time 90 minutes.

Questions

Section A: 2 marks per question

1. In a coordinate plane, let $A \equiv (0,0)$; and let points B to Z be arranged in a 5×5 grid formation as show in the diagram. Find the value of $AB^2 + AC^2 + AD^2 + \cdots + AY^2 + AZ^2$.



2. A bakery offers three flavors of cake. In how many ways can 5 people buy one cake each; such that no three of them buy exactly the same flavor?

3. Let ΔABC have side lengths $13, 14$ and 15. If I is the incenter, then find out the smallest area among the triangles $\Delta IBC, \Delta ICA$ and ΔIAB.

4. Find the smallest non-negative value of k such that the equations
$y = x^2 + 2x + 2$ and $y = kx - 2$ have atleast one common solution.

5. For any $n \in \mathbb{N}$, let $f(n)$ denote the number formed by writing the decimal representations of n and n^2, in that order from left to right.
For eg, $f(1) = 11, f(4) = 416, f(12) = 12144$.
If $K = f(1) + f(2) + f(3) + \cdots + f(19) + f(20)$; then find the sum of the digits of K.

Section B: 3 marks per question

6. Given ΔABC such that $\angle A = 90^\circ$ and $BC = 12$; let Γ be its circumcircle. Let the tangent to Γ at A meet line BC at point P such that $B - C - P$. Given that the angles of ΔABP are in arithmetic progression, find the area of ΔABC.

7. Find the number of ordered pairs of integers (m, n) such that
$1 \leq m \leq n \leq 30$ and the numbers $m^2, 2mn, 3n^2$ are in A.P.

8. For any natural number k, let $m(k)$ denote the largest digit in the decimal representation of k. For eg, $m(1) = 1, m(24) = 4, m(55) = 5$.
Find the value of $m(1) + m(2) + m(3) + \cdots + m(98) + m(99)$.

9. In a group of 10 students, there are 4 pairs of twins. The group wants to pose for a photo, in two distinct rows of 5 people each; such that each twin is adjacent to their sibling in the same row.
How many such distinct photos can be taken? (Assume that twins look perfectly identical in a photo!)

10. Find the number of ordered pairs of integers (a, b) such that
$-3 \leq a, b \leq 3$, and the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have at least one common root.

Section C: 5 marks per question

11. Let ΔABC such that $AB = 7, BC = 8, CA = 9$.
Let Γ_1 be a circle passing through A, and tangent to line BC at point B.
Let Γ_2 be a circle passing through A, and tangent to line BC at point C.
Let Γ_1, Γ_2 intersect again at P; and let the line AP intersect BC at K. Find the length of AK.

12. For any natural number n, let $L(n)$ denote the L.C.M. of n and 27. Find the number of distinct terms in the series $L(1), L(2), L(3), \cdots, L(26), L(27)$, if we count any repeated values only once.

13. Let $P(x) = x^2 - 6x + 10$ and $Q(x) = 2x^2 - 12x + 10$. Count the number of ordered pairs of integers (a, b) such that $P(a) > b > Q(a)$.

14. $ABCD$ is an isosceles trapezium such that $AB || CD$, $\angle ABC = 45^\circ$,
$AB = 2CD$ and $BC = AD = 6$. Count the number of points P which lie strictly in the interior of $\square ABCD$, such that the distance of P from lines BC, AD are both positive integers.

15. A *cross* is a tile of area 5, that looks like a plus sign; i.e. it has one center square, and 4 squares around the center square, one in each direction (left, right, up and down).
Given a blank chessboard of size 8×8, what is the minimum number of squares that need to be coloured, so that no matter where we place a cross, it will cover exactly one coloured square?
Note that the cross can only be placed along the grid lines (i.e. it cannot cover any square partially); and it should also completely stay within the board area (i.e. no part can hang outside the board)

Answers and hints

1. **[550]** Use the distance formula; separately add the X and Y components.

2. **[090]** Choose which flavor is not repeated (3 ways), then choose who selects that flavor (5 ways), finally partition the others into 2 distinct groups of 2 each (6 ways).

3. **[026]** By Heron's formula, $\Delta = 84 = rs$; so $r = 6$ which is the common height for all the three given triangles.

4. **[006]** By subtraction we get a quadratic equation whose discriminant $(2-k)^2 - 16$ should be non-negative. So $2-k \geq 4$ or $2-k \leq -4$; with the former impossible due to the non-negative condition of k.

5. **[020]** Partitioning 1 to 20 as per the number of digits in n^2, we get
$K = (1^2 + 2^2 + \cdots + 20^2) + 10 \times (1 + \cdots + 3) + 100 \times (4 + \cdots + 9) + 1000 \times (10 + \cdots + 20) = 171830.$

6. **[018]** Let $\angle ABP = x$; then the angles of ΔABP are $x, 90+x, 90-2x$. If these are in A.P, it forces $x = 15°$. If O is the midpoint of BC, then $AO = \frac{BC}{2} = 6$, and $\angle AOC = 2x = 30°$, so the height from A to BC is 3.

7. **[040]** $m^2 + 3n^2 = 4mn$ implies $(m-n)(m-3n) = 0$.
So either $m = n$ (30 solutions) or $m = 3n$ (10 solutions).

8. **[615]** For each i in 1 to 9, there are exactly $2i+1$ numbers n such that $f(n) = i$; namely $i0, i1, \cdots, i(i-1), 0i, 1i, \cdots, (i-1)i, ii$. So we need to find $\sum_{i=1}^{9}(i(2i+1)) = 2\sum_{i=1}^{9} i^2 + \sum_{i=1}^{9} i$.

9. **[432]** First partition the 4 twin pairs into the two rows (6 ways); then partition the 2 non-twins into those rows (2 ways); finally permute each row, treating the twin pairs as a single entity (6×6 ways)

10. **[013]** Subtracting, we get $x(a-b) = a-b$.
So either $a = b$ (7 solutions) or $x = 1$ implying $a + b = -1$ (6 solutions).

11. **[007]** By the power of K with respect to Γ_1, Γ_2, $KB^2 = KA \cdot KP = KC^2$; so AK is a median; and we can use Apollonius' theorem.

12. **[018]** Noting that $L(k) = L(3k)$ for any k in 1 to 27, we can partition 1 to 27 into groups ignoring any powers of 3 in their factorization.

13. **[030]** Note that $P(a) > Q(a)$ forces $0 < a < 6$. For each a in 1 to 5, we add up $P(a) - Q(a)$, subtracting 1 in each case, due to the strict bounds.

14. **[040]** Joining BC, AD at point O, we can setup a coordinate system with O as the origin, and BC, AD as the axes, so that A, B, C, D have coordinates $(0,12), (12,0), (6,0), (0,6)$ respectively. Now we only need to count the lattice points in the interior of $\square ABCD$.

15. **[012]** The only way to ensure that any cross covers exactly one colored square, is to use a 'knight's tour" layout for colouring, as shown below.