MPI RMO Selection test 2022

Each question has an integer answer from 000 to 999.
Maximum marks 50; total time 90 minutes.

Questions

Section A: 2 marks per question

1. For a regular 12-sided polygon, find the ratio of the lengths of its largest and smallest diagonals.

2. Find the number of ordered pairs of natural numbers (m, n) such that $m < n$ and the L.C.M. of m and n is 2022.

3. In ΔABC, $AB = 21, BC = 28$ and $\angle B = 90^\circ$. Find the radius of a circle whose center lies on seg AC, and which is tangent to AB and BC.

4. Find the number of three digit numbers which are divisible by 11, and whose digits are all odd numbers.

5. Find the value of $k > 0$ for which the equation $x^2 - kx + 2k + 2 = 0$ has roots α, β such that $|\alpha - \beta| = 1$.

Section B: 3 marks per question

6. A bakery offers 3 flavors of cake. In how many ways can 5 people buy one cake each, such that each flavor gets chosen by at least one person?

7. Find the sum of all $k \in \mathbb{R}$ such that the equations $(k-6)x + 2y = 2$ and $6x + (k-7)y = 3$ have no common solution for $x, y \in \mathbb{R}$.

8. Given integers a, b whose G.C.D. is 15, find the largest possible value of the G.C.D. of $3a + 5b$ and $3a - 5b$.

9. Find the area of a rhombus, given that the distance between its opposite sides is 12, and the sum of its diagonals is 35.

10. If $x, y \in \mathbb{Q}$ such that $x^2 + y^2 = \frac{5}{2}$ and $x^3 + y^3 = \frac{13}{4}$, then find $8(x + y)$.

Section C: 5 marks per question

11. Let Γ_1 be a circle with diameter $AB = 18$, and Γ_2 be the circle with center A and radius AB. Let Γ_3, Γ_4 be circles of equal radius r that are tangent to both Γ_1, Γ_2 and also tangent to each other. Find r.

12. A king can move exactly one square vertically, horizontally or diagonally. On a standard 8×8 chessboard, the distance between any two squares is defined as the minimum number of moves it takes for the king to travel from one square to the other. Find the number of ordered pairs of squares (x, y) such that the distance between x and y is exactly 7.

13. Let $a, b, c, d \in \mathbb{R}$ such that $a^2 + 4b^2 + c^2 + 4d^2 = 2(ab + bc + cd + da)$. Find the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

14. In ΔABC with $AB = 80$ and $AC = 64$, let D be the midpoint of side BC. Let E, F be points on sides CA, AB respectively such that $\square ABDE$ and $\square CAFD$ are cyclic. Let the internal bisector of $\angle FDE$ meet EF at point K. If $FK = 16$, find KE.

15. Alice plays the following game, starting with an even number M written on the board: In each move, she divides the number on the board by 2, then adds 5 to the quotient, and writes this answer, erasing the previous number. The game ends when the answer is an odd number. (For example, if $M = 50$, the game ends at the 3rd move: $50 \to 30 \to 20 \to 15$)
Find the smallest value of M for which the game ends at the 5th move.

Answer key and hints

1. **[002]** It is enough to consider the regular hexagon formed by alternate vertices of the 12-gon.
2. **[013]** We can write $m = 2^{a_1} \times 3^{a_2} \times 337^{a_3}$, $n = 2^{b_1} \times 3^{b_2} \times 337^{b_3}$ with each pair $(a_i, b_i) \in \{(0,1),(1,0),(1,1)\}$, but not all pairs can be $(1,1)$.
3. **[012]** If the desired radius is r, then the area of ΔABC is $\frac{(AB+BC)r}{2}$.
4. **[010]** If the number is $(abc)_{10}$ then $a + c = b + 11 \in \{12, 14, 16, 18\}$; and each possibility can be counted separately.
5. **[009]** If the discriminant is Δ then $|\alpha - \beta| = \sqrt{\Delta}$; hence $k^2 - 8k - 8 = 1$.
6. **[150]** Consider one case where some flavor gets chosen by three people, and another case where two flavors get chosen by two people each.
7. **[003]** Eliminating x we get $(k-10)(k-3)y = 3(k-10)$; which has no solution only if $k = 3$. (If $k = 10$ there are infinitely many solutions.)
8. **[450]** $\text{GCD}(3a+5b, 3a-5b) \leq \text{GCD}(6a, 10b) = 2\text{GCD}(3a, 5b)$.
   The GCD of $3a, 5b$ can be at most $3 \times 5 = 15$ times that of a, b.
9. **[150]** If diagonals are $2x, 2y$, side is z and area of the rhombus is S then $x + y = \frac{35}{2}$, $xy = \frac{S}{2} = 6z$ and $z^2 = x^2 + y^2 = (\frac{35}{2})^2 - 12z$.
10. **[008]** If $s = x + y, p = xy$ then $x^2 + y^2 = s^2 - 2p, x^3 + y^3 = s(s^2 - 3p)$.
    Eliminating p gives a cubic in s with $s = 1$ being the only rational root.
11. **[008]** Let O_1, O_3 be the centers of Γ_1, Γ_3, let T be the point of tangency of Γ_3 and Γ_4, and let $AT = h$. By Pythagoras theorem in ΔAO_3T and ΔO_1O_3T, $h^2 + r^2 = (18-r)^2$ and $(9+h)^2 + r^2 = (9+r)^2$.
12. **[252]** Each corner square is at a distance of 7 from all squares of the two opposite edges i.e. 15 squares, and all the other edge squares is at a distance of 7 from all squares of the opposite one edge i.e. 8 squares.
13. **[005]** Multiplying by 2 and rearranging, we get $(a-2b)^2 + (2b-c)^2 + (c-2d)^2 + (2d-a)^2 = 0$ which implies $a = 2b = c = 2d$.
14. **[025]** $\Delta ABC \sim \Delta DBF \sim \Delta DEC$ implies $\frac{AC}{AB} = \frac{DF}{DB}$ and $\frac{AB}{AC} = \frac{DE}{DC}$; hence $\frac{FK}{KE} = \frac{DF}{DE} = (\frac{AC}{AB})^2$.
15. **[042]** If the answer after t moves is x_t, then $x_0 = M$ and $x_{t+1} = \frac{x_t+10}{2}$; which implies $x_{t+1} - 10 = \frac{x_t - 10}{2}$ and $x_t = \frac{M-10}{2^t} + 10$. If T is the highest power of 2 that divides $M - 10$, the game ends after exactly T moves.