MPI RMO Selection test 2023

Each question has an integer answer from 000 to 999.
Maximum marks 50; total time 90 minutes.

Questions

Section A: 2 marks per question

1. Consider all triples of real numbers (x, y, z) such that $x + 2y + 2z = 11$ and $2x + y - 5z = 10$. Find the smallest possible value of $x^2 + y^2$.

2. A pyramid consists of a square base and four other sides which are conguent isosceles triangles. Given that the side length of the square base is 12, and the vertical height of the pyramid is 18, find the radius of the smallest sphere that can fully contain the pyramid.

3. Find the largest possible value of $\mathrm{GCD}(63 + n, 62 - n)$ where $n \in \mathbb{N}$ such that $1 \le n \le 60$.

4. Seven schools attend an event, with two students participating from each school. In how many ways can we select a team of five students such that no two of them belong to the same school?

5. Find the sum of all $n \in \mathbb{N}$ such that $n^3 + 13n^2 + 95n + 83$ is the perfect cube of an integer.

Section B: 3 marks per question

6. Given ΔABC such that $\angle C = 90°$, $AC = 15$ and $BC = 20$; let a circle centered at C be tangent to seg AB at R, and intersect segments CA, CB at P, Q respectively. If the area of $\square CPRQ$ is expressed as $\frac{x}{y}$ where x, y are coprime positive integers, then find the value of $x + y$.

7. Positive real numbers x, y, z_1, z_2 satisfy the following properties:

   - x, y, z_1 are in Arithmetic Progression, and x, y, z_2 are in Geometric Progression, in that order.
   - $x + y = 30$ and $z_2 - z_1 = 16$.

   Find the value of $z_1 + z_2$.

8. Given a board of 16 unit squares arranged in a 4×4 grid, count the number of rectangles formed whose length and breadth are of opposite parity (i.e. one is odd and one is even).

9. Consider all quadratic equations of the form $x^2 + bx + c = 0$ where b, c are real numbers, and the equation has distinct real roots with a difference of 4 betweeen them. Find the smallest possible value of $4b + 4c + 20$.

10. Given a regular pentagon $ABCDE$, let l be a line through A parallel to CD. Let the line BC intersect the lines l and DE in P and Q respectively. Find the value of $\dfrac{PQ^2}{QC^2}$.

Section C: 5 marks per question

11. Let $S = \{(m, n) : m, n \in \mathbb{Z}; 2m^2 = 3n^2 - 1\}$. Let $a, b, c, d \in \mathbb{N}$ such that:

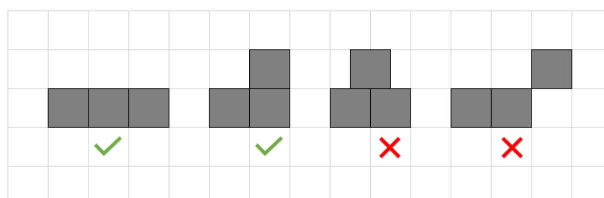
    - $2ab = 3cd$
    - For all $(m, n) \in S$, we also have $(am + bn, cm + dn) \in S$.

    Find the smallest possible value of $a + b + c + d$.

12. We are given five square tiles to form various shapes as per the following rules:

    - Tiles should be placed next to each other, so that they all form a single connected shape.
    - Adjacent tiles should be aligned perfectly, so that the vertices of the common sides coincide.
    - Two shapes are identical if one shape can be rotated in the plane to form the other one.

    Here are some examples of valid and invalid shapes that can be formed using three tiles:

    Find the number of distinct shapes that can be formed using all the five tiles as per the above rules.

13. Find the smallest possible value of $\left(64x + \dfrac{9}{1 + 16x}\right)$ over all positive real values of x.

14. Given an acute ΔABC with orthocenter H and circumcenter O, let the line AH meet the circumcircle of ΔABC again at a point D. If $AD = 7$, and the distance of O from the internal bisector of $\angle BAC$ is 1, then find the circumradius of ΔABC.

15. Let $n \in \mathbb{N}$ such that $1 \leq n \leq 99$. Let $ABCD$ be a rectangular room such that $AB = CD = 100$ and $BC = AD = 50$. Let P be a point on seg CD such that $DP = n$. A ray of light starts at vertex A and travels in the direction of P. After one or more bounces on the walls (which are all perfect mirrors), the ray eventually reaches one of the corners of the room, and stops there. Find the number of values of n for which the ray stops at C.

Answer key and hints

1. **[025]** Eliminating z, we get $3x+4y=25$ which implies $(x-3)^2+(y-4)^2 = x^2+y^2-25 \geq 0$ with equality at $(x,y)=(3,4)$.

2. **[011]** It suffices to find the circumradius of the isosceles triangle formed by one diagonal of the base square (with length $12\sqrt{2}$) and the apex of the pyramid (with height 18 from the base).

3. **[025]** Any common divisor of $63+n$ and $62-n$ will also divide their sum 125; so given the constraints on n, the G.C.D. can be atmost 25 when $n=12$ or $n=37$.

4. **[672]** We first select 5 schools in $\binom{7}{5}$ ways, and one student from each school in 2^5 ways.

5. **[010]** The given expression lies between $(n+4)^3$ and $(n+6)^3$; hence it must be equal to $(n+5)^3$.

6. **[509]** Since $CR \perp AB$, we have $CP = CQ = CR = 12$ and $AR = 9, RB = 16$; and we can use similar triangles to find the distances of points P and Q from line CR.

7. **[082]** Using $z_1 = 2y-x$ and $z_2 = \frac{y^2}{x}$, we obtain a quadratic equation in x with solutions $x=9$ or $x=25$, out of which the latter yields a negative value for z_1.

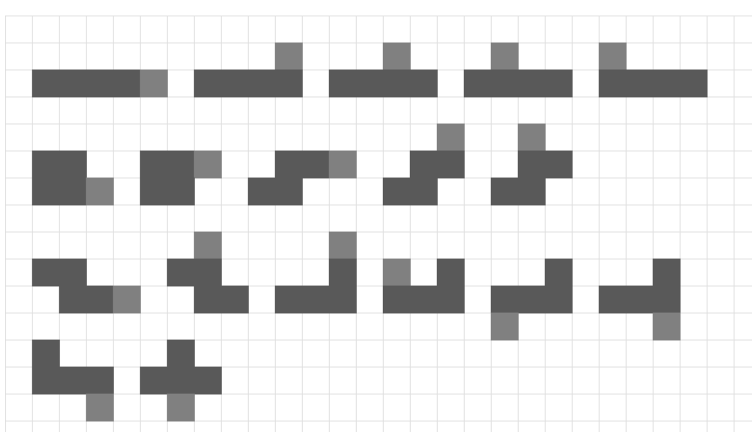
8. **[048]** One pair of sides can be chosen in 6 ways while the other pair can be chosen in 4 ways; and vice-versa depending on whether the length or the breadth is odd.

9. **[000]** The given condition implies $b^2-4c=16$. If $b+c=k$, eliminating c we get $4k+20=(b+2)^2 \geq 0$ with equality for $b=-2, c=-3, k=-5$.

10. **[004]** Let the lines l and DE intersect at R. We can find various angles to deduce that ΔPAB and ΔRAE are isosceles, so $PA = AB = CD = AE = AR$. Hence in ΔPQR, $CD||PR$ and $CD = \frac{1}{2}PR$, hence C is the midpoint of seg PQ.

11. **[020]** If $2m^2 = 3n^2-1$ and $2(am+bn)^2 = 3(cm+dn)^2-1$, we get $m^2(2a^2-3c^2-2) = n^2(3d^2-2b^2-3)$. This forces the terms inside brackets to be zero, since otherwise the ratio $m^2 : n^2$ would be constant for all $(m,n) \in S$, which is not true. The smallest possible value of $a+b+c+d$ is realized by the solution $a=5, b=6, c=4, d=5$.

12. **[018]** The set of all distinct shapes can be constructed by adding one square at a time, and discarding duplicates under rotation. After obtaining all the possible shapes using four squares, the result of adding the fifth square is illustrated here:



13. **[008]** The given expression can be rearranged as $6\left(\frac{32x+2}{3}+\frac{3}{32x+2}\right)-4$, wherein the value inside the bracket is the sum of reciprocals, and reaches its minimum value of 2 when $\frac{32x+2}{3}=1$ i.e. $x=\frac{1}{32}$.

14. **[004]** Let the circumradius of ΔABC be r; and let the internal, external bisectors of $\angle BAC$ intersect the circumcircle of ΔABC in L, K resp. Then KL is a diameter, and $KLDA$ is an isosceles trapezium, with $AK = DL = 2$, and $AL = DK = \sqrt{4r^2-4}$. So by Ptolemy's theorem we get $4r^2 - 14r - 8 = 0$.

15. **[012]** We set up a 2D coordinate system such that $A \equiv (0,0), B \equiv (100,0), D \equiv (0,50), P \equiv (n,50)$. Then the ray AP will pass through some point $K \equiv (100x, 50y)$ where x, y are coprime positive integers. Segment AK represents the light's path if it travelled without reflecting on the walls (illustrated below for $n = 40$). In particular, the ball stops at C if and only if x, y are both odd. $A - P - K$ implies $100x = ny$; so x, y are both odd if and only if n is divisible by 4 but not by 8.