PRMO 2018: Practice paper I: Hints and Answers

Maharashtra and Goa Region

Questions

Q1 Count the number of 3-digit natural numbers N with the property that the sum of the digits of N is divisible by the product of the digits of N.

Q2 Let x, y be positive real numbers such that:
$\frac{x}{y} + \frac{y}{x} = \frac{5}{2}$, and $\frac{x^2}{y} + \frac{y^2}{x} = \frac{27}{8}$. Find the value of $4x + 4y$.

Q3 Let C_1, C_2 be orthogonal circles of radii $6, 3$ respectively. If one of their external common tangents touches the two circles in points A, B, then find the length of seg AB.

Q4 Let $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that $f(1) = g(1) = 1$,
$f(n+1) - f(n) = g(n)$, and $g(n+1) - g(n) = f(n+1)$ for all $n \in \mathbb{N}$.
How many even numbers are there in the set $\{f(1), f(2), \cdots, f(100)\}$?

Q5 Find the number of ordered pairs (p, q) such that p, q are both prime numbers less than 50, and $pq + 1$ is divisible by 12.

Q6 The distance between cities P and Q is 60 kilometers. Two buses continuously travel back and forth between the two cities, from 6AM to 6PM.
At 6AM, the first bus starts from P and runs at 25 km/h; while the second bus starts from Q and runs at 45 km/h. How many times will the two buses cross each other, during their operation period of 6AM to 6PM?
(Assume that the buses travel continuously at constant speeds, without any stopping or turning time at the ends.)

Q7 In ΔABC, let D be the midpoint of side BC; and let P be the midpoint of AD. Let the lines BP, CP intersect the opposite sides AC, AB in points E, F respectively. If the area of $\square AFPE$ is 2, then find the area of ΔABC.

Q8 Let function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n \times n!$ for all $n \in \mathbb{N}$. Determine the largest prime factor of $1 + f(1) + f(2) + f(3) + \cdots + f(30)$.

Q9 Let $\square ABCD$ be a parallelogram such that a circle of radius 6 is inscribed inside it (i.e. circle is tangent to all four sides). If the diagonals of $\square ABCD$ are in the ratio $3 : 4$, then find its perimeter.

Q10 We have 4 pencils, each of a different colour (say R,Y,G,B). Similarly we have 4 erasers, each of a different colour (the same 4 colours: R,Y,G,B). Let N be the number of ways in which 4 students can be given 1 pencil and 1 eraser each, so that no student gets both items of the same colour. What is the sum of digits of N?

Q11 $\square ABCD$ is a cyclic quadrilateral, whose diagonals are perpendicular to each other; and are of length 6 and 8 respectively. Determine the smallest possible radius of the circumcircle of the quadrilateral.

Q12 There is a 4×4 board whose squares are all white. What is the maximum number of squares that can be painted black, without forming a 1×3 horizontal or vertical strip of black squares?

Q13 In ΔABC, $\angle A = 100^\circ$, $\angle C = 30^\circ$. Point D is taken on seg AC such that $\angle ADB = 50^\circ$. If $AD = 4$ and $AB = 6$, then find AC.

Q14 Consider the expression $f(n) = n^{99} + 2n^{98} + 3n^{97} + \cdots + 98n^2 + 99n + 100$. What is the remainder of $f(2)$ on dividing by 100?

Q15 How many 4-digit numbers can be formed using only the digits 1 and 2 (with repetitions allowed) that are divisible by 11?

Q16 Find the smallest integer k such that for any $n > 2$, for any n positive reals $a_1, a_2, \cdots, a_n$, we will have $\dfrac{(\sum i a_i)^2}{\sum a_i^2} \leq 8kn^3$. (The summations are taken over all i from 1 to n.)

Q17 Let ΔABC be a triangle with integer sides, and its area be k. Let H be its orthocenter; and M, N be the midpoints of seg BC and AH respectively. If $k \times MN = 6$, then find the perimeter of ΔABC.

Q18 Let x, y be positive real numbers which satisfy the equations: $x^2 + x = y + 5$ and $x^3 = x^2 y + 4$. Find the value of $x^3 - y^3$.

Q19 Let C_1, C_2 be two circles such that the radius of C_1 is four times that of C_2; and the circles are externally tangent at point P. Let line l be their common tangent at point P; and let l intersect one of their external common tangents at point K. If $KP = 6$, then find the distance between the centres of the two circles.

Q20 Let $A_1 A_2 A_3 \cdots A_9$ be a regular 9-gon. How many acute-angled triangles are formed by these 9 vertices?

Q21 Given that the polynomials $P(x) = x^3 - 3x - 2$ and $Q(x) = x^3 - 5x^2 + 8x - 4$ have a common root; find the difference between the largest root of P and the smallest root of Q.

Q22 An ice-cream vendor uses cones whose vertical height is 3 times the radius of the circular top. He fills the entire cone with icecream, plus an additional scoop that is a perfect hemisphere on the top (i.e. half of a sphere, with the same radius as the top.) If the total volume of the ice-cream is $\frac{110}{21}$, then what is the radius of the top of the cone? (Assume $\pi \approx \frac{22}{7}$)

Q23 Let x, y, z be positive integers such that $xy + yz = 2xz$. If $\sqrt[x]{10} = \sqrt[y]{20} = \sqrt[z]{N}$, then find the value of N.

Q24 We have an infinite supply of tiles of size 8×6. What is the smallest integer N such that a floor of size $60N \times 7N$ can be fully covered with such tiles?
(Tiles can only be placed horizontally or vertically; cannot overlap or be broken)

Q25 Consider $P(x) = 5x^6 - ax^4 - bx^3 - cx^2 - dx - 9$, where $a, b, c, d \in \mathbb{R}$. If the roots of $P(x)$ are in AP, find the value of a.

Q26 A team of N workers starts building a wall. At the end of each hour, one worker permanently leaves the team. As a result, the wall was completed in 16 hours. But if all workers had kept working (i.e. nobody left) throughout the task, then the wall could have been completed in only 12 hours. Find the value of N.

Q27 ABC is an equilateral triangle of side length 6. Let P and Q be arbitrary points taken on the sides AB and AC respectively, such that $PQ = 2$. Let K be the midpoint of seg PQ. Find the minimum possible value of $AK^2 + BK^2 + CK^2$, over all possible choices for the positions of P and Q.

Q28 Let $A_1, A_2, \cdots, A_{10}$ be 10 points chosen on a circle. In how many ways can one select a set of 10 chords of the form A_iA_j such that no subset of the chosen chords would form a closed loop (i.e. a polygon of any size)?

Q29 Find the number of solutions (m, n) to the equation $m^2(m+1)^2 + 5n = 1000000$, where m, n are non-negative integers.

Q30 In a standard clock, let O denote the center of the clock, and A denote the position of the 12-hour mark; and at any given time, let M, H represent the positions of the minute and hour hands, around the circumference of the clock. Then in the six-hour duration of 3PM to 9PM, how many times does ray OM coincide with the interior angle bisector of $\angle AOH$?

Hints

1. If the digits are a, b, c in some order, such that c is the largest; then it means $abc \leq a + b + c \leq 3c$.
2. Convert all given information in terms of the basic symmetric expressions $(x+y)$ and (xy).
3. Just a repeated application of Pythagoras Theorem
4. List the first few terms of f, g and guess the odd/even pattern before proving it.
5. All primes greater than 3 are congruent to either $+1$ or -1 modulo 6.
6. Draw a rough diagram of the paths of both buses.
7. Use Ceva's and Menelaus' theorems to determine unknown ratios.
8. Evaluate the expression $f(1) + f(2) + \cdots$ as a telescopic series
9. If a circle is inscribed inside a parallelogram, the parallelogram is actually a rhombus.
10. This is closely related to the standard problem of derangements.
11. We need to find the smallest circle that can accomodate the given diagonals as chords.
12. Subdivide the board suitably in order to use Pigeonhole Principle.
13. SImple angle-chasing leads to similarity.
14. This is a standard Arithmetico-Geometric Progression
15. Make cases based on the possible sums of alternate digits.
16. Application of Cauchy-Schwartz inequality, followed by estimation.
17. Use standard formulae for triangle area.
18. Solve for x^2 and $x - y$.
19. What is the relation between K and the points of tangency?
20. Count by fixing one vertex at a time.
21. The common root of P and Q will also be a root of $P(x) - Q(x) = 0$.
22. Direct application of standard formulae.
23. x, y, z are in Harmonic Progression.
24. Look at the terms in the prime factorization of the floor dimensions.
25. The coeffients give us the sum and product of the roots, from which we can find their value.

26. Restate the given information in terms of the fraction of work completed per-hour per-worker.
27. The given expression is directly related to the distance of K from the centroid of the triangle.
28. Try for smaller polygons to make a conjecture before proving it.
29. Analyze the equation modulo 5.
30. Just count the number of occurrences in each clock hour.

Answers

Q	A	Q	A	Q	A
1	10	11	04	21	01
2	09	12	11	22	01
3	06	13	09	23	40
4	33	14	74	24	06
5	84	15	19	25	09
6	09	16	05	26	30
7	12	17	09	27	45
8	31	18	07	28	00
9	50	19	15	29	13
10	09	20	30	30	06