PRMO 2019: Practice paper I

Maharashtra and Goa Region

Time: 3 hours 28 July 2019

Instructions

- The answer to each question is a positive integer from 00 to 99. Students should mark their response on the OMR sheet by darkening the bubbles for the corresponding tens' place and units' place digits.
- No other markings should be made on the answer sheet. Blank sheets will be provided for rough work.
- The use of calculators of any kind is not allowed.

Questions

Q1 Given the polynomials $P(x) = 2x^2 + 5x - 2$ and $Q(x) = 9x + 7$, let S be the set of all $x \in \mathbb{R}$ such that $P(x) = Q(x)$. Find the sum of all values in S.

Q2 Let a, b be positive real numbers such that $a^2b + ab^2 = 2.5$ and $a^3b + ab^3 = 4.25$. Find the value of $ab + a^2b^2$.

Q3 An ice-cream shop offers 6 different flavors. N students go to the shop, and each student buys 3 flavors of icecream. Given that no two students have 2 or more flavors in common, what is the maximum possible value of N?

Q4 Find the number of pairs (x, y) of natural numbers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2019}$.

Q5 Let ABC be a piece of cardboard shaped like an equilateral triangle of side 6, and let O be its centroid. From the cardboard, we cut out and remove the circular sector BOC. If the perimeter of the remaining shape is expressed as $a + b\pi$ where a, b are positive integers, find the value of $a + b$.

Q6 There is a circular train track of circumference 60 km, with stations A and B located at diametrically opposite positions. At 6 AM, a train leaves from station A and travels clockwise at 20 km/h. Also at 6 AM, another train leaves from B and travels anti-clockwise at 40 km/h. Both trains keep travelling non-stop at a constant speed around the circular track, until 6 PM. How many times did the two trains cross each other?

Q7 Let $P(x)$ be a polynomial whose coefficients are all positive integers. Given that $P(1) = 9$ and $P(20) = 1662$, find $P(3)$.

Q8 Let x and y be real numbers such that $1 < x \le y \le 10$. Find the maximum possible value of $\dfrac{y - |y - x|}{y^3 - 2y^2 + 3y - x}$.

Q9 Let $\square ABCD$ be a parallelogram such that a circle of radius 6 is inscribed inside it (i.e. circle is tangent to all four sides). If the diagonals of $\square ABCD$ are in the ratio $3:4$, then find its perimeter.

Q10 Find the smallest positive real number k such that $a+b+c+kabc \geq 8\sqrt{abc}$ for any positive real numbers a, b, c.

Q11 Consider $P(x) = 5x^6 - ax^4 - bx^3 - cx^2 - dx - 9$, where $a, b, c, d, e \in \mathbb{R}$. If the roots of $P(x)$ are in AP, find the value of $5a$.

Q12 In how many ways can you arrange 3 red, 2 blue and 2 green coins in a straight line, so that no two red coins are adjacent?

Q13 In ΔABC, let D be the midpoint of side BC; and let P be the midpoint of AD. Let the lines BP, CP intersect the opposite sides AC, AB in points E, F respectively. If the area of $\square AFPE$ is 2, then find the area of ΔABC.

Q14 In ΔABC, $\angle A = 100°$, $\angle C = 30°$. Point D is taken on seg AC such that $\angle ADB = 50°$. If $AD = 4$ and $AB = 6$, then find AC.

Q15 Four towns A, B, C, D are such that there are exactly two roads between each pair of towns. If a traveller starts at town A, wants to visit each town exactly once, and then return to A again; in how many ways can he plan his journey?

Q16 $\square ABCD$ is a cyclic quadrilateral, whose diagonals are perpendicular to each other; and are of length 6 and 8 respectively. Determine the smallest possible radius of the circumcircle of the quadrilateral.

Q17 A team of N workers starts building a wall. At the end of each hour, one worker permanently leaves the team. As a result, the wall was completed in 16 hours. But if all workers had kept working (i.e. nobody left) throughout the task, then the wall could have been completed in only 12 hours. Find the value of N.

Q18 Given ΔABC with $\angle A = 60°, AB = 0.5, AC = 0.8$. If the length of the altitude from A to side BC is h, then find the value of $49h^2$.

Q19 A circular table has 6 identical chairs around it. Find the number of ways in which 2 teachers and 4 students can be seated, such that no 2 teachers are adjacent.

Q20 Let m, n be distinct natural numbers such that $m+i | n+i$ for each $i = 0, 1, 2, 3, 4$. Find the smallest possible value of $m + n$.

Q21 There is a 4×4 board whose squares are all white. What is the maximum number of squares that can be painted black, without forming a 1×3 horizontal or vertical strip of black squares?

Q22 Let $A_1, A_2, \cdots, A_8$ be 8 points chosen on a circle. In how many ways can one select a set of 4 chords of the form A_iA_j such that they would form a closed loop?

Q23 Let function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n \times n!$ for all $n \in \mathbb{N}$; and let $g(n) = f(n+1) - f(n)$. Determine the remainder of $1 + g(1) + g(2) + g(3) + \cdots + g(49)$ on dividing by 50.

Q24 For any positive integer n, let T_n denote the sum of numbers $1, 2, 3, \cdots, n$. Find the number of solutions (m, n) to the equation $T_m - T_n = 9$, where m, n are non-negative integers.

Q25 Let C_1, C_2 be two circles such that the radius of C_1 is four times that of C_2; and the circles are externally tangent at point P. Let line l be their common tangent at point P; and let l intersect one of their external common tangents at point K. If $KP = 6$, then find the distance between the centres of the two circles.

Q26 Consider the expression $f(n) = n^{99} + 2n^{98} + 3n^{97} + \cdots + 98n^2 + 99n + 100$. What is the remainder of $f(2)$ on dividing by 100?

Q27 Let x, y be positive real numbers which satisfy the equations: $x^2 + x = y + 5$ and $x^3 = x^2y + 4$. Find the value of $x^3 - y^3$.

Q28 Find the number of solutions (m, n) to the equation $m^2(m+1)^2 + 5n = 1000000$, where m, n are non-negative integers.

Q29 Let $ABCD$ be a square with side length 8. Let P, Q, R, S be points on sides AB, BC, CD, DA respectively such that $PQRS$ is a rectangle of area 32. Find the length of seg PR

Q30 Given that the polynomials $P(x) = x^3 - 3x - 2$ and $Q(x) = x^3 - 5x^2 + 8x - 4$ have a common root; find the difference between the largest root of P and the smallest root of Q.