PRMO 2019: Practice paper I: Hints and Answers

Maharashtra and Goa Region

Time: 3 hours | 28 July 2019

Questions

Q1 Given the polynomials $P(x) = 2x^2 + 5x - 2$ and $Q(x) = 9x + 7$, let S be the set of all $x \in \mathbb{R}$ such that $P(x) = Q(x)$. Find the sum of all values in S.

Answer: 02

Hint: S is essentially the set of all roots of the equation $P(x) - Q(x) = 0$ which is a quadritic equation with positive discriminant. Also note that it is not necessary to find the actual roots, we only need to know the sum of the roots, which can be determined directly from the coefficients.

Q2 Let a, b be positive real numbers such that $a^2b + ab^2 = 2.5$ and $a^3b + ab^3 = 4.25$. Find the value of $ab + a^2b^2$.

Answer: 02

Hint: Rewrite the equations in terms of variables $s = a + b$ and $p = ab$. Eliminate s to get a cubic equation for p, for which it is easy to guess the root $p = 1$.

Q3 An ice-cream shop offers 6 different flavors. N students go to the shop, and each student buys 3 flavors of icecream. Given that no two students have 2 or more flavors in common, what is the maximum possible value of N?

Answer: 04

Hint: From 6 flavours, one can list $\binom{6}{2} = 15$ distinct pairs of flavours; and when each student chooses 3 flavours, he picks $\binom{3}{2} = 3$ pairs from the list. Thus we can have max 5 students; further elimination by counting the pairs w.r.t. to a single flavour (say flavor #1).

Q4 Find the number of pairs (x, y) of natural numbers such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2019}$.

Answer: 09

Hint: Since both x, y are greater than 2019, let $x = 2019+x_1$ and $y = 2019+y_1$. By simplification we get $x_1y_1 = 2019^2$ which has 9 positive solutions, each leading to a different solution to the original equation.

Q5 Let ABC be a piece of cardboard shaped like an equilateral triangle of side 6, and let O be its centroid. From the cardboard, we cut out and remove the circular sector BOC. If the perimeter of the remaining shape is expressed as $a + b\pi$ where a, b are positive integers, find the value of $a + b$.

Answer: N/A

Hint: The radius of the sector being removed is $2\sqrt{3}$; and central angle is $120°$. The length of the circular arc BOC is $4\frac{\pi}{\sqrt{3}}$. So the perimeter is $12 + 4\frac{\pi}{\sqrt{3}}$. *This cannot be expressed as $a+b\pi$ for integers a, b, so the question is incorrect.*

Q6 There is a circular train track of circumference 60 km, with stations A and B located at diametrically opposite positions. At 6 AM, a train leaves from station A and travels clockwise at 20 km/h. Also at 6 AM, another train leaves from B and travels anti-clockwise at 40 km/h. Both trains keep travelling non-stop at a constant speed around the circular track, until 6 PM. How many times did the two trains cross each other?

Answer: 12

Hint: The relative speed of the trains towards each other is 60 km/h, so the trains cross each other once every 12 hours.

Q7 Let $P(x)$ be a polynomial whose coefficients are all positive integers. Given that $P(1) = 9$ and $P(20) = 1662$, find $P(3)$.

Answer: 47

Hint: From $P(1)$ we see that all coefficients are less than 20; hence $P(20)$ can be interpreted as an evaluation of the polynomial coefficients as digits in base 20; giving the polynomial $P(x) = 4x^2 + 3x + 2$.

Q8 Let x and y be real numbers such that $1 < x \leq y \leq 10$. Find the maximum possible value of $\frac{y - |y - x|}{y^3 - 2y^2 + 3y - x}$.

Answer: 01

Hint: Eliminating the absolute value sign, and noting that increasing x will increase the value of the expression, we replace x by y; and by simplification we get $\frac{1}{(y-1)^2+1}$.

Q9 Let $\square ABCD$ be a parallelogram such that a circle of radius 6 is inscribed inside it (i.e. circle is tangent to all four sides). If the diagonals of $\square ABCD$ are in the ratio $3 : 4$, then find its perimeter.

Answer: 50

Hint: $ABCD$ must be a rhombus; and let its diagonals intersect at O. Then in right-angled ΔAOB with height 6, we can find the arm lengths having ratio $3 : 4$.

Q10 Find the smallest positive real number k such that $a + b + c + kabc \geq 8\sqrt{abc}$ for any positive real numbers a, b, c.

Answer: 16

Hint: AM-GM with $a, b, c, kabc$

Q11 Consider $P(x) = 5x^6 - ax^4 - bx^3 - cx^2 - dx - 9$, where $a, b, c, d, e \in \mathbb{R}$. If the roots of $P(x)$ are in AP, find the value of $5a$.

Answer: 35

Hint: The roots in A.P. have sum 0, hence they must be of the form $-5d, -3d, -d, d, +3d, +5d$ for some real number d.

Q12 In how many ways can you arrange 3 red, 2 blue and 2 green coins in a straight line, so that no two red coins are adjacent?

Answer: 60

Hint: First arrange the 2 blue and 2 green coins in $\binom{4}{2} = 6$ ways, and choose with of the 5 empty spaces to be filled with one each of the 3 red coins.

Q13 In ΔABC, let D be the midpoint of side BC; and let P be the midpoint of AD. Let the lines BP, CP intersect the opposite sides AC, AB in points E, F respectively. If the area of $\square AFPE$ is 2, then find the area of ΔABC.

Answer: 12

Hint: By Ceva's theorem etc, we get $AE : EC = AF : FB = 1 : 2$, from which we get the areas of $\Delta ABP, ABD$ etc.

Q14 In ΔABC, $\angle A = 100°$, $\angle C = 30°$. Point D is taken on seg AC such that $\angle ADB = 50°$. If $AD = 4$ and $AB = 6$, then find AC.

Answer: 09

Hint: Note that $\Delta ABC \sim \Delta ADB$

Q15 Four towns A, B, C, D are such that there are exactly two roads between each pair of towns. If a traveller starts at town A, wants to visit each town exactly once, and then return to A again; in how many ways can he plan his journey?

Answer: 96

Hint: The order of visiting towns can be chosen in $3! = 6$ ways, and in each case one has $2^4 = 16$ choices for the roads to be used.

Q16 $\square ABCD$ is a cyclic quadrilateral, whose diagonals are perpendicular to each other; and are of length 6 and 8 respectively. Determine the smallest possible radius of the circumcircle of the quadrilateral.

Answer: 04

Hint: The diameter of the circle cannot be smaller than the larger diagonal of the quadrilateral.

Q17 A team of N workers starts building a wall. At the end of each hour, one worker permanently leaves the team. As a result, the wall was completed in 16 hours. But if all workers had kept working (i.e. nobody left) throughout the task, then the wall could have been completed in only 12 hours. Find the value of N.

Answer: 30

Hint: Let each worker complete x fraction of the wall in 1 hour; then we have $Nx = \frac{1}{12}$ and $Nx + (N-1)x + \cdots + (N-15)x = 1$.

Q18 Given ΔABC with $\angle A = 60°, AB = 0.5, AC = 0.8$. If the length of the altitude from A to side BC is h, then find the value of $49h^2$.

Answer: 12

Hint: By cosine rule, $BC = 0.7$; and the area of ΔABC is $0.1\sqrt{3}$

Q19 A circular table has 6 identical chairs around it. Find the number of ways in which 2 teachers and 4 students can be seated, such that no 2 teachers are adjacent.

Answer: 72

Hint: First we seat one student; then the other students can be arranged in 3! unique ways. Then the teachers can be placed in any 2 of the 4 gaps which are now uniquely labelled with respect to the position of the first student.

Q20 Let m, n be distinct natural numbers such that $m+i|n+i$ for each $i = 0, 1, 2, 3, 4$. Find the smallest possible value of $m + n$.

Answer: 62

Hint: Some trial and elimination based on LCM of 2,3,4,5,6

Q21 There is a 4×4 board whose squares are all white. What is the maximum number of squares that can be painted black, without forming a 1×3 horizontal or vertical strip of black squares?

Answer: 11

Hint: Divide the board into 5 strips of 3 each, plus one extra square.

Q22 Let $A_1, A_2, \cdots, A_8$ be 8 points chosen on a circle. In how many ways can one select a set of 4 chords of the form A_iA_j such that they would form a closed loop?

Answer: 70

Hint: Choosing any four vertices leads to a unique closed non-intersecting loop. (Note that the question did not mention non-intersecting which was a mistake.)

Q23 Let function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n \times n!$ for all $n \in \mathbb{N}$; and let $g(n) = f(n+1) - f(n)$. Determine the remainder of $1 + g(1) + g(2) + g(3) + \cdots + g(49)$ on dividing by 50.

Answer: 00

Hint: Note that $f(n) = (n + 1)! - n!$ and $g(n) = (n + 2)! - 2(n + 1)! + n!$. The given sum can be converted into a telescopic series.

Q24 For any positive integer n, let T_n denote the sum of numbers $1, 2, 3, \cdots, n$. Find the number of solutions (m, n) to the equation $T_m - T_n = 9$, where m, n are non-negative integers.

Answer: 03

Hint: Noting that $T_n = \frac{n(n+1)}{2}$, the given equation can be converted into $(2m+1)^2 - (2n+1)^2 = 72$. There are 3 possible factorizatons of 72 which leads to odd solutions for $x^2 - y^2 = 72$.

Q25 Let C_1, C_2 be two circles such that the radius of C_1 is four times that of C_2; and the circles are externally tangent at point P. Let line l be their common tangent at point P; and let l intersect one of their external common tangents at point K. If $KP = 6$, then find the distance between the centres of the two circles.

Answer: 15

Hint: If the radii are r and $4r$, then the length of the external common tangent is $4r = 12$; giving $r = 3$.

Q26 Consider the expression $f(n) = n^{99} + 2n^{98} + 3n^{97} + \cdots + 98n^2 + 99n + 100$. What is the remainder of $f(2)$ on dividing by 100?

Answer: 74

Hint: Writing $f(2)$ as an arithmetico-geometric progression, we simplify $f(2)$; and find its remainders modulo 4 and 25; and use Chinese Remainder Theorem

Q27 Let x, y be positive real numbers which satisfy the equations: $x^2 + x = y + 5$ and $x^3 = x^2y + 4$. Find the value of $x^3 - y^3$.

Answer: 07

Hint: Convert the equations into new variables $p = x^2$ and $q = x - y$.

Q28 Find the number of solutions (m, n) to the equation $m^2(m+1)^2 + 5n = 1000000$, where m, n are non-negative integers.

Answer: 13

Hint: Use modulo 5 along with simple bounds to narrow the possible values of m

Q29 Let $ABCD$ be a square with side length 8. Let P, Q, R, S be points on sides AB, BC, CD, DA respectively such that $PQRS$ is a rectangle of area 32. Find the length of seg PR

Answer: 08

Hint: Observe that the sides of $PQRS$ must be parallel to the diagonals of $ABCD$, thus forming isosceles right triangles on all sides.

Q30 Given that the polynomials $P(x) = x^3 - 3x - 2$ and $Q(x) = x^3 - 5x^2 + 8x - 4$ have a common root; find the difference between the largest root of P and the smallest root of Q.

Answer: 01

Hint: To find the common root, consider the roots of $P(x) - Q(x) = 0$.