IOQM 2023 practice test

Each question has an integer answer from 00 to 99.
Maximum marks 50; total time 90 minutes.

Section A: 2 marks per question

1. Let $k \in \mathbb{R}$ such that the equations $x^3 + 2x^2 + (k-9)x - (k+2) = 0$ and $x^2 - 7x + (k+6) = 0$ have a common root α. If S is the sum of all the possible values of α, then find $10S$.

2. Consider ΔABC such that $AB = 13$, $BC = 14$, $CA = 15$. Let E be the midpoint of seg AC, and let point K lie on seg BE such that $AK \perp BC$. If AK is expressed as $\frac{m}{n}$ where $m, n \in \mathbb{N}$ are coprime, then find the sum of the digits of $m + n$.

3. For any $m, n \in \mathbb{N}$ let $g(m, n) = \text{GCD}(m^2, mn, n^2)$. Find the largest possible value of $g(m, n)$ where $m, n \in \mathbb{N}$ such that $10 < m < n < 20$.

4. Let N be the number of ways to distribute 10 distinct pens and 5 distinct pencils into 3 identical boxes with 5 objects in each box, so that each box has more pens than pencils. Find the sum of the digits of N.

5. Let $ABCD$ be a square of side length 2. Let P, Q be the midpoints of sides AB, BC respectively. Let $PD \cap AQ = X$, $PC \cap AQ = Y$, $PC \cap DQ = Z$. If the area of $\square DXYZ$ is expressed as $\frac{m}{n}$ where $m, n \in \mathbb{N}$ are coprime, then find $m + n$.

Section B: 3 marks per question

6. Let $a_1, a_2, a_3, \cdots$ be a sequence of rational numbers such that $a_1 = \frac{1}{3}$, and $a_{n+1} = \frac{1}{1+a_n}$ for all $n \geq 1$. If $a_{100} = \frac{p}{q}$ where $p, q \in \mathbb{N}$ are coprime, find the remainder obtained upon dividing $p + q$ by 12.

7. Let $S = \{1, 2, 3, \cdots, 99, 100\}$. Find the smallest $n \in \mathbb{N}$ such that for any subset $T \subseteq S$ with $|T| = n$, there exist $a, b, c \in T$ such that $a < b < c$, and a is a factor of b, and b is a factor of c.

8. On a standard 8×8 chessboard, let N be the number of ways in which one rook and one bishop can be placed so that neither piece attacks the other one. Find the sum of the digits of N. (A rook attacks vertically and horizontally, while a bishop attacks diagonally.)

9. Let Γ be a circle of radius 363, with diameter AB and centre O. Let Ω be a circle of radius r such that Ω is internally tangent to Γ, and Ω is also tangent to seg AB at a point T. Given that $r \in \mathbb{N}$ and $OT \in \mathbb{N}$, find the number of distinct possible values of r.

10. Find the number of ordered 6-tuples (a, b, c, d, e, f) of natural numbers such that:
    - a, b, c are in A.P. and $a < b < c$.
    - d, e, f are in A.P. and $d < e < f$.
    - ad, be, cf are also in A.P. and $ad < be < cf$.

Section C: 5 marks per question

11. Let S be the number of ordered pairs (m, n) of natural numbers such that m, n are both divisors of 900, and m, n are coprime. Find the sum of the digits of S.

12. Consider ΔABC with H as its orthocentre. Let X be the midpoint of seg AH, Q be the midpoint of seg AC, and F be the foot of the altitude from C to line AB. Given that ΔFQX is equilateral, find the value of $\angle BAC$ in degrees.

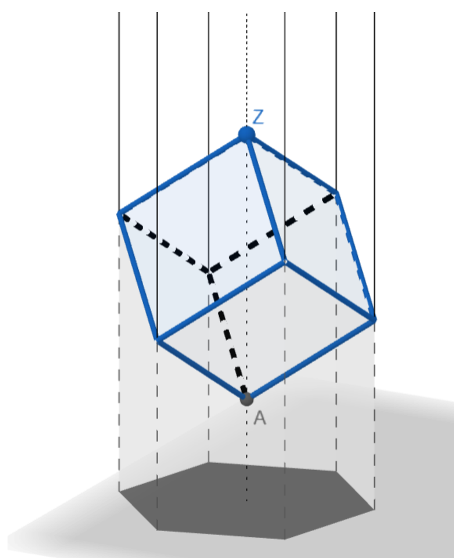
13. Let $k \in \mathbb{N}$ such that $k < 89$. Consider the sequence of integers defined by: $a_1 = 89, a_2 = k$, and $a_{n+2} = |a_{n+1} - a_n|$ for all $n \geq 1$. Let $z(k)$ denote the smallest index $t \in \mathbb{N}$ such that $a_t = 0$. Find the smallest possible value of $z(k)$.

14. In a party of 15 people, each person has atleast 1 friend and atmost 13 friends. Friendship is a relation with the following properties for any three distinct persons A, B, C:

    - If A is a friend of B, then B is also a friend of A.
    - If A, B are friends, and B, C are friends, then A, C are also friends.

    If N is the number of ordered pairs (A, B) of persons such that A and B are friends, then find the difference between the largest and smallest possible values of N.

15. Consider a wooden cube of side length 3, and let A, Z denote one pair of opposite vertices. As denoted in the illustration, suppose the cube is (magically) hovering in air such that line AZ is perpendicular to the ground, while the sun is also shining directly overhead. If the area of the shadow cast by the cube on the ground can be written as $S\sqrt{3}$, find the value of S.



Answer key and hints

1. **[22]** By subtracting the equations repeatedly we get $48x = 10k + 56$, and now eliminate k.

2. **[16]** If $AK \cap BC = D$, and the midpoint of DC is M, then we can find $BD = 5$, $DC = 9$, $AD = 12$ and $EM = 6$. Now use B.P.T in ΔBEM to find $KD = \frac{60}{19}$.

3. **[36]** Note that $g(m, n) = \text{GCD}(m, n)^2$, and for any divisor greater than 6, there don't exist two distinct multiples in the range $(11, 20)$.

4. **[09]** The only valid distribution of pens is $4-3-3$, which can be done in $\binom{10}{4}\binom{6}{3}/2$ ways. Now the boxes can be considered distinct, and the pencils can be distributed in $5\binom{3}{2}$ ways.

5. **[31]** $\square DXYZ$ is a kite, so it suffices to find DY and XZ. Note that $D - Y - B$ and by similarity $DY = \frac{2}{3}DB = \frac{4\sqrt{2}}{3}$. Also $XZ||PQ$, so $\frac{XZ}{PQ} = \frac{DX}{DP}$. Using $AX \perp DP$ we can find $XZ = \frac{4\sqrt{2}}{5}$.

6. **[06]** If each $a_n = \frac{p_n}{q_n}$ where $p_n, q_n \in \mathbb{N}$ are coprime, then $p_{n+1} = q_n$ and $q_{n+1} = p_n + q_n$, since these equations imply that p_{n+1}, q_{n+1} are also coprime. We observe that $p_n + q_n$ modulo 3 (resp. 4) goes through a cycle with period 8 (resp. 6), so modulo 12 the cycle has period 24.

7. **[76]** For any T if such $a, b, c \in T$ exist then $c \geq 2b \geq 4a$. So for the 75-element subset $T = \{26, 27, 28, \cdots, 99, 100\}$, such a, b, c, cannot exist. Further, consider sets $S_1, S_3, S_5 \cdots$ such that for each odd number $i < 100$, $S_i = \{x \in S | x = i \cdot 2^j \text{ for some } j \in \mathbb{N}\}$. Then one can show that if $|T| \geq 76$, there exist $a < b < c \in T$ belonging to the same S_i, hence $a|b$ and $b|c$.

8. **[20]** The board can be divided into concentric rings, such that if the rook is placed anywhere on the outermost ring, then the bishop has 42 valid positions; and this number keeps decreasing by 2 for each subsequent inner ring. The total possibilities are $28 \times 42 + 20 \times 40 + 12 \times 38 + 4 \times 36$.

9. **[05]** Using Pythagoras theorem we get $OT = 11\sqrt{33^2 - 6r}$, so it suffices to consider all the perfect squares that $33^2 - 6r$ can take; which are also constrained to be odd multiples of 3.

10. **[00]** If the first two A.P.s are $b - x, b, b + x$ and $e - y, e, e + y$, then $xy = 0$.

11. **[08]** Let t denote the number of distinct prime factors of m; we can make cases based on t. For eg, if $t = 1$, and $2|m$ then m has 2 possibilities, and n can be any divisor of $3^2 \cdot 5^2$. Similarly if $t = 2$ and $2 \cdot 3|m$ then m has 4 possibilities, and n can be any divisor of 5^2.

12. **[30]** $QX||HC$ and $AH = 2FX = 2QX = HC$ implies $AB = BC$. Further one can deduce that $\angle ABC$ is obtuse, hence $C - F - H$ and $60° = \angle XQF = \angle QFC = \angle QCF = \angle ACF$.

13. **[12]** Since 89 is prime, if $a_t = 0$ then $a_{t-2} = a_{t-1} = 1$ is forced. Now one can work backwards and observe that 89 is a Fibonacci number.

14. **[140]** As per the given conditions, the party can be split into groups of people who are all each others' friends. The possible sizes of the groups with the smallest and the largest values of N is (2,2,2,2,2,2,3) and (13,2) respectively.

15. **[09]** We observe that the shadow is a regular hexagon, and joining its alternate vertices forms an equilateral triangle, whose side length equals the diagonal of any face of the cube, namely $3\sqrt{2}$.