# Regional Mathematical Olympiad - 2016: Practice paper
## Hints and Solutions

Hints/Partial solutions:

1. (16 marks) Find all positive integers $m, n$ such that $2^m + 2016 = 2^n$.
   **Hint:**
   Rearranging the terms, we get the equivalent equation: $2016 = 2^n - 2^m$

2. (16 marks) Given $\Delta ABC$, let $P$ and $Q$ be the feet of the perpendiculars drawn from point $A$ onto the internal bisectors of angles $B$ and $C$ respectively. Prove that $PQ$ is parallel to $BC$.
   **Hint:**
   Let the internal bisectors of angles $B$ and $C$ meet at $I$, which is the incenter of $\Delta ABC$.

3. (16 marks) Find all real numbers $x, y, z$ that satisfy the following system of equations:
   $6x - 5y = xy$
   $6y - 5z = yz$
   $6z - 5x = zx$
   **Hint:**
   First, we consider the case that any one of $x, y, z$ may be zero.

4. (16 marks) There is a round table with 18 identical chairs around it. Find the number of ways in which 5 teachers and 13 students can be seated, so that no 4 students are sitting together. (Note that any two seating arrangements which are identical by rotation, will be considered identical)
   **Hint:**
   In any given valid seating configuration, let us denote the teachers as $T_1, T_2, \cdots, T_5$, in the order in which they appear clockwise, starting from any one teacher.
   Let $x_1$ denote the number of students seated directly to the left of $T_1$ (i.e. in between $T_1$ and $T_2$). Similarly we define $x_2, x_3, \cdots, x_5$.

5. (16 marks) Given a fixed segment $BC$ and a line $l$ parallel to it, find with proof the position of a point $A$ on line $l$, for which the measure of $\angle BAC$ is the maximum.
   **Hint:**
   We note that the triangle's area $\Delta = \frac{1}{2}bc\sin A$ is constant, since its base and height are of a constant length.
   By cosine rule: $\cos A = \frac{b^2+c^2-a^2}{2bc}$

6. (20 marks) Find the number of integers $m$ such that **exactly one** of the roots of the following quadratic equation is an integer:
   $$2x^2 - mx + 125 = 0$$

   **Hint:** Using the standard formula, we get the roots of the equation as $\alpha = \frac{m+\sqrt{\Delta}}{4}$; $\beta = \frac{m-\sqrt{\Delta}}{4}$
   with the discriminant as $\Delta = m^2 - 1000$, which has to be a perfect square, for any one of the above roots to be an integer.
   Let $\Delta = k^2$, where $k$ is a non-negative integer. Then the above roots are: $\alpha = \frac{m+k}{4}$; $\beta = \frac{m-k}{4}$.

Complete solutions:

1. (16 marks) Find all positive integers $m, n$ such that $2^m + 2016 = 2^n$.
**Solution:**
Rearranging the terms, we get the equivalent equation: $2016 = 2^n - 2^m = 2^m(2^{n-m} - 1)$
Since $m < n$, i.e. $n - m > 0$, the term inside the bracket is necessarily odd; so $2^m$ must be equal to the largest power of 2 that divides 2016. Based on the factorization of 2016, this forces $m = 5$; which gives $(m, n) = (5, 11)$ as the only solution.

2. (16 marks) Given $\Delta ABC$, let $P$ and $Q$ be the feet of the perpendiculars drawn from point $A$ onto the internal bisectors of angles $B$ and $C$ respectively. Prove that $PQ$ is parallel to $BC$.
**Solution:**
Let the internal bisectors of angles $B$ and $C$ meet at $I$, which is the incenter of $\Delta ABC$. Then we observe that $AIPQ$ is a cyclic quadrilateral, since are both right angles.
Hence $\angle QPB = \angle QPI = \angle QAI$
$= \angle QAC - \angle IAC = (90 - \frac{C}{2}) - \frac{A}{2} = \frac{B}{2} = \angle PBC.$
Therefore $PQ$ is parallel to $BC$.

3. (16 marks) Find all real numbers $x, y, z$ that satisfy the following system of equations:
$6x - 5y = xy$
$6y - 5z = yz$
$6z - 5x = zx$
**Solution:**
First, we consider the case that any one of $x, y, z$ may be zero; without loss of generality let $x = 0$. Substituting this in the first equation, we get $y = 0$, and in the third equation, we get $z = 0$.
Hence, if any one of $x, y, z$ is zero, then all of them are forced to be zero, and we get $(x, y, z) = (0, 0, 0)$ as one solution, which indeed satisfies the above system of equations.

Now, let us consider the case that none of $x, y, z$ is zero. Then we can define $a = \frac{1}{x}$, $b = \frac{1}{y}$, $c = \frac{1}{z}$.
By dividing each of the above equations by their RHS, we can rewrite the same as follows:
$6b - 5a = 1$
$6c - 5b = 1$
$6a - 5c = 1$
This is a linear system of three equations and three unknowns, and can be solved by several well known methods, such as Cramer's rule, or repeated elimination. By any method, we get the only solution as $(a, b, c) = (1, 1, 1)$; which gives us $(x, y, z) = (1, 1, 1)$.

Thus the entire solution set is $\{(0, 0, 0), (1, 1, 1)\}$.

4. (16 marks) There is a round table with 18 identical chairs around it. Find the number of ways in which 5 teachers and 13 students can be seated, so that no 4 students are sitting together. (Note that any two seating arrangements which are identical by rotation, will be considered identical)
**Solution:**
In any given valid seating configuration, let us denote the teachers as $T_1, T_2, \cdots, T_5$, in the order in which they appear clockwise, starting from any one teacher.
Let $x_1$ denote the number of students seated directly to the left of $T_1$ (i.e. in between $T_1$ and $T_2$). Similarly we define $x_2, x_3, \cdots, x_5$. Then, we know that $x_1 + x_2 + x_3 + x_4 + x_5 = 13$, with all $x_i < 4$. There are only three solutions to this integer equation (considering cyclic rotations to be identical): $(3, 3, 3, 3, 1)$, $(3, 3, 3, 2, 2)$ and $(3, 3, 2, 3, 2)$. Each solution corresponds to a specific configuration of chairs being reserved for students and teachers.
Also, it is important to note that all the three configurations are rotationally asymmetric; or in other words, once any configuration (say 3,3,3,3,1) has been chosen, each chair is effectively being given a distinct 'label' based on its position (for instance, 'three chairs to the right of the 1').
So for each of those three configurations, there are exactly 5! ways to arrange the teachers, and 13! ways to arrange the students, among their reserved chair types.
Hence the final answer is $3 \times 5! \times 13!$

5. (16 marks) Given a fixed segment $BC$ and a line $l$ parallel to it, find with proof the position of a point $A$ on line $l$, for which the measure of $\angle BAC$ is the maximum.
   **Solution:**
   We note that the triangle's area $\Delta = \frac{1}{2}bc\sin A$ is constant, since its base and height are of a constant length. By cosine rule: $\cos A = \frac{b^2+c^2-a^2}{2bc}$, or equivalently: $\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{4\Delta}$ ..(1)
   Since cot is a decreasing function in the $(0,\pi)$ range, so to maximize the value of $A$, we need to minimize the value of $\cot A$. On the RHS of the above equation (1), since $a$ and $\Delta$ are constant, so we want to minimize $b^2+c^2$.

   If $M$ is the midpoint of side $BC$, then by Apollonius' theorem, $b^2+c^2 = \frac{1}{2}a^2 + \frac{1}{2}AM^2$.
   So we need to find the position of $A$ that minimizes the distance $AM$.
   So the optimal position of $A$ would be at the foot of the perpendicular from $M$ onto the line $l$, which would cause $\Delta ABC$ to be an isosceles triangle.

6. (20 marks) Find the number of integers $m$ such that **exactly one** of the roots of the following quadratic equation is an integer:
   $$2x^2 - mx + 125 = 0$$

   **Solution:**
   Using the standard formula, we get the roots of the equation as $\alpha = \frac{m+\sqrt{\Delta}}{4}$; $\beta = \frac{m-\sqrt{\Delta}}{4}$
   with the discriminant as $\Delta = m^2 - 1000$, which has to be a perfect square, for any one of the above roots to be an integer.
   Let $\Delta = k^2$, where $k$ is a non-negative integer. Then the above roots are: $\alpha = \frac{m+k}{4}$; $\beta = \frac{m-k}{4}$.

   Claim: A necessary and sufficient condition for exactly one of the above roots to be an integer, is that both $m$ and $k$ are odd.
   Proof: The difference between the above two numerators is $2k$; so if $k$ were even, then both the numerators would be congruent modulo 4; and it would be impossible for exactly one of them to be divisible by 4. Hence $k$ is odd. Now, no matter which of the above two numerators is divisible by 4, it forces $m$ to be odd as well.
   Conversely as well, if $m, k$ are both odd, then it forces $m+k$ and $m-k$ to be congruent to 0 and 2 (modulo 4), in some order; meaning that exactly one of $\alpha, \beta$ is an integer.

   Hence, all we need is to find the number of odd pairs $(m, k)$ which satisfy the discriminant $k^2 = m^2 - 1000$; which is equivalent to $1000 = (m+k)(m-k)$.
   Let $m = 2m_1 + 1$; and $k = 2k_1 + 1$, where $m_1, k_1 \in \mathbb{Z}$. Then we get $250 = (m_1 + k_1 + 1)(m_1 - k_1)$.
   Using any integer factorization $250 = pq$, we get $m_1 = \frac{p+q-1}{2}$ and $k_1 = \frac{p-q-1}{2}$. Importantly, we note that $p$ and $q$ would always be of opposite parity, so $m_1, k_1$ would always be integers.
   Finally, the number of positive divisors of $250 = 2^1 5^3$ is $(1+1) \times (3+1) = 8$; so the total number of integer factorizations of the form $250 = pq$ would be twice as many, considering positive as well as negative divisor pairs.
   However, we note that swapping the value of $p$ and $q$ does not change the value of $m_1$, and hence the value of $m$ as well. So we also have to divide by 2, to eliminate the double counting of multiple factorizations which are yielding the same value of $m$.

   So our final answer is that there are 8 possible values of $m$, that satisfy the original condition of the problem.