Geometry problems

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We use the following common notation for all problems:
In ΔABC with orthocenter H, centroid G; let P, Q, R be the midpoints of the sides BC, CA, AB; D, E, F be the feet of the altitudes onto the sides BC, CA, AB, and X, Y, Z be the midpoints of AH, BH, CH respectively.
Let t_a, t_b, t_c denote the lengths of the internal angle bisectors from points A, B, C respectively.

Level 1

E1 In ΔABC with orthocenter H, let P, Q, R be the midpoints of the sides; D, E, F be the feet of the altitudes, and X, Y, Z be the midpoints of AH, BH, CH respectively. Prove that these nine points are concyclic, using a different starting point than ΔDEF.

- (a) Or in other words, prove that $PQRD, PQRE, PQRF$ are cyclic, as well as $PQRX, PQRY, PQRZ$.
- (b) Similarly, prove that $XYZD, XYZE, XYZF$ are cyclic, as well as $XYZP, XYZQ, XYZR$.

E2 Find the general expression for t_a in terms of the side lengths a, b, c.

- (a) Prove that $b = c$ if and only if $t_b = t_c$.
- (b) Given ΔABC with side lengths $a = 4, b = 5, c = 6$, let CD be the internal angle bisector. Prove that $CD = AD$.

E3 Prove that the triangles ABC, BHC, CHA, AHB all have the same circumradius.

Level 2

M1 Construct ΔABC, given the lengths R, a, t_a.

M2 Extend EF, let it meet line BC at point K. Then prove that ΔAPK has orthocenter H.

M3
- (a) Given ΔABC with side lengths $a = 4, b = 5, c = 6$, let CD be the internal angle bisector. Prove that $CD = AD$.
- (b) Find the general condition which the sides a, b, c should satisfy, for $CD = AD$ to be true.