Pre-RMO Practice test 1

30 questions · 30 marks · 3 hours

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August 12, 2017

Instructions

- The answer of each question is in the form of a number in the range 00 to 99. You are expected to darken the corresponding digits bubble in the answer sheet, for each question. (The upper row of $0-9$ digits is the tens' place, and the lower row is the units' place.)
- There are 30 questions, each worth 1 mark. There is no negative marking. Maximum marks are 30.
- You can do rough work in your notebook.

Questions

1. Consider the equation $3m^2+5n=1402$ where m,n are positive integers. Find the sum of all the possible values of m which satisfy this equation.
2. In a rectangle $ABCD$, let P be a point on side AB such that $CP \perp DP$. The circumcircles of ΔDAP and ΔCBP intersect a second time at point K. Given that $CP = 15$ and $DP = 20$, what is the value of KP?
3. If m,n are integers that satisfy the equation $4m^2 - 6mn + 4m = 6n - 9n^2 + 6mn + 24$, then find the smallest possible value of $m^2 + n^2$.
4. 9 lines are constructed in the plane, in 3 groups of 3 lines, such that the 3 lines within each group are all parallel to each other (but not parallel to any lines from the other groups). What is the total number of parallelograms formed?
5. Let Δ_n denote the sum of the first n natural numbers. For example, $\Delta_5 = 1+2+3+4+5 = 15$. If k is a positive integer such that $\Delta_k \leq 2017 < \Delta_{k+1}$, then what is the largest prime factor of k?
6. What is the largest two-digit integer n such that the ratio $\frac{LCM[n,588]}{GCD(n,588)}$ is a perfect square?
7. A circle Γ_1 is constructed with diameter AB of length 16. Point P is taken on segment AB such that $AP = 3PB$. A circle Γ_2 is constructed such that it is tangent to seg AB at point P, and also internally tangent to Γ_1 at a point Q. Then find the radius of circle Γ_2.
8. Find the sum of all two-digit natural numbers n such that $n^2 + 129$ is a perfect square.
9. The combined score of 5 batsmen in a game is 140 runs. If the lowest individual score is 25 runs, then what is the largest possible value of the highest individual score?
10. Let x, y be real numbers such that $7x - 24y = 25$. What is the minimum possible value of $x^2 + y^2$?
11. Let $S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$. What is the value of $\sum_{n=1}^{10} \frac{1}{1-S_n}$?
12. In how many ways can we tile a 10×10 floor using tiles of size 4×1? (Tiles cannot be cut into smaller pieces, and should not overlap with each other)

13. Let n be the largest integer such that the product of any 7 odd consecutive numbers (such as $1, 3, 5, 7, 9, 11, 13$) is always divisible by n. Then what is the sum of the digits of n?

14. Let x_1, x_2 be the roots of the equation $x^2 - 3x + 1 = 0$.
What is the value of the expression $\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)$?

15. Circles Γ_1 and Γ_2 have centers O_1 and O_2 respectively, and radii equal to 5 units and 3 units respectively. Segment AB is an external common tangent to the two circles, with point A lying on Γ_1 and B lying on Γ_2. Point P lies on seg AB such that $AP = 2PB$. Then what is the value of $O_1P^2 - 4O_2P^2$?

16. Let N be the smallest natural number such that the L.C.M. of N^3 and $8!$ is a perfect square. What is the sum of the digits of N?

17. In ΔABC, $\angle A = 60°$ and the inradius $r = 2$. The internal bisector of $\angle A$ meets the circumcircle of ABC a second time at point L. If the distance of L from line BC is $\frac{3}{2}$, then find the square of the perimeter of ΔABC.

18. $a_1, a_2, \cdots, a_n, \cdots$ is a sequence of positive integers such that $a_n = 3a_{n-1} + 2a_{n-2}$. If $a_1 = 12$, what is the smallest possible value of a_2 such that the G.C.D. of a_{100} and a_{102} is 4?

19. In an isosceles triangle ABC with $AB = AC = 5$, the internal bisector of $\angle A$ meets side BC at point K. Point L is taken on line AC such that $A - C - L$ and $BC = LC$. What is the value of $AK \times AL$?

20. k is a positive integer such that the equations $x^2 + kx - 18 = 0$ and $2x^2 - 5x - k = 0$ have a common root. If this common root is an integer, find the value of k.

21. In a 10-sided polygon $A_1A_2 \cdots A_{10}$, we construct all diagonals of the form A_iA_j where i and j are of the same parity (i.e. both odd or both even). Find the number of points of intersection of these diagonals (Don't include the original 10 vertices. Also, assume that no three diagonals are concurrent)

22. Let $P(x)$ be a cubic polynomial with integer coefficients, such that $P(n)$ is divisible by n^2 for each positive integer n. If $P(1) = 3$ and $P(2) = 20$, then what is the value of $P(3)$?

23. Two circles have centers O_1 and O_2, and radii 4 and 5 respectively. They intersect in points A and B; such that $O_1A \perp O_2A$. If the external common tangent touches the two circles at points P and Q, then what is PQ^2?

24. 36 students are sitting in an 6×6 grid fashion. If each student shakes hands with each of their adjacent neighbours (i.e. left, right, front, back), then what is the total number of handshakes?

25. Let N be the number of 4-letter words can be formed using the letters from the word 'MATHEMATICS', such that it contains 2 consonants and 2 vowels. Find the sum of the digits of N.

26. Let $k = \cos^2(54°) - \cos^2(36°)$. What is the value of $4k^2 - 2k + 1$?

27. In how many ways can you put 15 identical balls into 3 distinct boxes, such that each box contains at least 1 and at most 10 balls?

28. Find the largest positive integer N with the following property: From a shop that sells chocolates of N different brands, if we buy any 125 chocolates, then we will surely have bought at least 4 chocolates of the same brand. (Assume that the shop has an infinite supply of all the brands)

29. In ΔABC, $\angle A = 60°$ and $\angle B = 45°$. G and H are the centroid and the orthocenter respectively, and R is the circumradius. If $GH^2 = kR^2$ then find the value of $(9k - 8)^2$?

30. Each square of a 2×3 board is to be colored white or black. Two coloring patterns are considered identical if one can be rotated through a 180° to form the other one. Then how many distinct coloring patterns are there?