1. Throughout this document, it is assumed that we are considering points in the real plane; and we follow the convention of using lowercase symbols to denote the complex coordinates of the corresponding points.

2. There is a natural bijection between the set of all 5-tuples of points in the real plane, and the set $\mathbb{C}^5$. For simplicity, let us refer to any such 5-tuple as well as its representation in $\mathbb{C}^5$ as a *pentagon*.

3. For any three points $A, B, C$ such that $A \neq B$, there exists a unique $s \in \mathbb{C}$ such that $c = (1-s)a + sb$. Further, starting from any other points $A', B'$ we can construct a unique point $C'$ with coordinate $c' = (1-s)a' + sb'$, such that $\Delta ABC \sim \Delta A'B'C'$. Let us refer to $C'$ as the *projection* of $(A', B')$ under the shape $s$.

4. Given a pentagon $Z = (Z_1, Z_2, Z_3, Z_4, Z_5)$ and $s \in \mathbb{C}$, we can construct a new pentagon $Z' = (Z'_1, Z'_2, Z'_3, Z'_4, Z'_5)$ where cyclically each $Z'_i$ is the projection of $(Z_i, Z_{i+1})$ under the shape $s$. Let us refer to $Z'$ as the *projection* of $Z$ under the shape $s$. Further, if we consider the elements of $\mathbb{C}^5$ as column vectors, then this transformation can be rephrased as a matrix equation $Z' = \mathcal{P}_s Z$, where $\mathcal{P}_s$ is the *projection matrix* of $s$, defined as follows:

$$\mathcal{P}_s = \begin{bmatrix} 1-s & s & 0 & 0 & 0 \\ 0 & 1-s & s & 0 & 0 \\ 0 & 0 & 1-s & s & 0 \\ 0 & 0 & 0 & 1-s & s \\ s & 0 & 0 & 0 & 1-s \end{bmatrix}$$

5. We observe that $\sum z'_i = \sum(sz_i + (1-s)z_{i+1}) = \sum z_i$; hence the pentagons $Z$ and $Z'$ have the same centroid. Henceforth, we shall restrict our discussion to only the pentagons whose centroid is the origin; i.e. $\sum z_i = 0$. For clarity, let us denote the set of such *origin-centered* pentagons as $\mathbb{C}^5_0$.

6. We also observe that all the projection matrices of the above family commute with each other. Hence, if we successively apply multiple shapes to any pentagon, the order in which we apply the shapes does not matter.

7. Let $s_1, s_2, s_3 \in \mathbb{C}$. Then, starting with an arbitrary pentagon $Z \in \mathbb{C}^5_0$, we can successively apply the shapes $s_1, s_2, s_3$ to construct new pentagons; which can be expressed as $Z' = P_{s_1} Z$; $Z'' = P_{s_2} Z'$ and $Z''' = P_{s_3} Z''$; or equivalenty $Z''' = P_{s_3} P_{s_2} P_{s_1} Z$. As noted above, all the projection matrices commute with each other; hence the order of shapes $s_1, s_2, s_3$ can be freely permuted, without causing any change to the final pentagon.

8. **Problem 1:** Determine all $s_1, s_2, s_3 \in \mathbb{C}$ with the following property: For any starting pentagon $Z \in \mathbb{C}^5_0$, the final pentagon $Z''' = P_{s_3} P_{s_2} P_{s_1} Z$ is always a regular pentagon.

9. The necessary and sufficient condition for an origin-centered pentagon to be regular, can be expressed as follows: Rotating the pentagon around the origin through an angle of $2\pi/5$ radians should be equivalent to a cyclic shift of all the vertices. (We are making assumptions about the order and orientation of the vertices; but with a suitable reordering of the vertices, this condition can be applied to any pentagon without loss of generality.)

10. To express the above condition formally, consider $\beta = \cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})$; and let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ denote the *rotation matrix*, the *shift matrix* and the *test matrix* respectively, defined as:

$$\mathcal{R} = \begin{bmatrix} \beta & 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & \beta \end{bmatrix}; \mathcal{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \mathcal{T} = \mathcal{R} - \mathcal{S} = \begin{bmatrix} \beta & -1 & 0 & 0 & 0 \\ 0 & \beta & -1 & 0 & 0 \\ 0 & 0 & \beta & -1 & 0 \\ 0 & 0 & 0 & \beta & -1 \\ -1 & 0 & 0 & 0 & \beta \end{bmatrix}$$

Then for any $Z \in \mathbb{C}^5_0$, the necessary and sufficient condition for $Z$ being a regular pentagon is $\mathcal{R}Z = \mathcal{S}Z$; or equivalently, $\mathcal{T}Z = 0$. Hence, the problem can be rephrased as:

11. **Problem 2:** Determine all $s_1, s_2, s_3 \in \mathbb{C}$ such that for any $Z \in \mathbb{C}^5_0$, we have $\mathcal{T} P_{s_3} P_{s_2} P_{s_1} Z = 0$.
By considering special values for $Z$ (for example, setting any two components as $+1$ and $-1$, and all others as zero), and other simplifications, one can reduce the problem further to:
**Problem 3:** Determine all $s_1, s_2, s_3 \in \mathbb{C}$ such that $(I - S)\mathcal{T} P_{s_3} P_{s_2} P_{s_1} = 0$.

12. Assuming all $s_i \neq 1$, we can rewrite all the components in terms of the identity matrix $I$ and the shift matrix $S$: $P_s = s(S + \frac{1-s}{s} I)$ and $\mathcal{T} = -(S + (-\beta)I)$. Introducing some substitutions for the ease of simplification:

$$k = s_1 s_2 s_3; t_i = \frac{1 - s_i}{s_i}; t_4 = -\beta; t_5 = -1$$

Noting that all the elements involved commute with each other, the expression from problem 3 can be expanded:

$$
\begin{aligned}
(I - S)\mathcal{T} P_{s_3} P_{s_2} P_{s_1} &= k(S + t_1 I)(S + t_2 I)(S + t_3 I)(S + t_4 I)(S + t_5 I) \\
&= k(S^5 + \sigma_1 S^4 + \sigma_2 S^3 + \sigma_3 S^2 + \sigma_4 S + \sigma_5 I) \\
&= k(\sigma_1 S^4 + \sigma_2 S^3 + \sigma_3 S^2 + \sigma_4 S + (\sigma_5 + 1)I)
\end{aligned}
$$

Here $\sigma_1$ to $\sigma_5$ are defined as the elementary symmetric expressions of $t_i$'s. Equating the above expression to the zero matrix, and noting that the various powers of $S$ refer to non-overlapping elements of the matrices, the condition of problem 3 is equivalent to $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$ and $\sigma_5 = -1$. The only possible solution is that $t_i$'s are equal to $-\beta, -\beta^2, -\beta^3, -\beta^4, -\beta^5 = -1$ in some order.

13. Since $t_4 = -\beta$ and $t_5 = -1$, this implies $\{t_1, t_2, t_3\} = \{-\beta^2, -\beta^3, -\beta^4\}$. Translating this back into the original $s_i$'s, the shapes $s_1, s_2, s_3$ correspond to constructing isosceles triangles on the sides of the pentagons, with apex angles of $72°, 144°$ and $216°$ respectively.