Problem

Let p be a prime number. Consider the set $\mathbb{P} = \{0, 1, \cdots, p^2 + p\}$.
Construct a set $\mathbb{L} = \{L_0, L_1, \cdots, L_{p^2+p}\}$ where each L_i is a subset of $\mathbb{P}$, such that:

1. For each $L_i \in \mathbb{L}$, $|L_i| = p + 1$;
2. For each $i \in \mathbb{P}$, there are exactly $p + 1$ sets from $\mathbb{L}$ that contain i;
3. For each distinct $L_i, L_j \in \mathbb{L}$, $|L_i \cap L_j| = 1$;
4. For each distinct $i, j \in \mathbb{P}$, there is exactly one set from $\mathbb{L}$ that contains both i and j.

Solution

We introduce the following notation:
For each $q \in \{0, 1, 2, \cdots, p-1\}$, let $A_q = \{qp + r | 0 \leq r < p\}$; and let $B = \{p^2, p^2 + 1, \cdots, p^2 + p\}$.
Clearly, $A_0, A_1, \cdots, A_{p-1}, B$ form a disjoint partition of $\mathbb{P}$; and each A_q is a complete set of residues modulo p.
For any $q \in \{0, 1, 2, \cdots, p-1\}$ and any $n \in \mathbb{Z}$; let $r_q(n)$ denote the unique element $d \in A_q$ such that $n \equiv d (\mod p)$.

We claim that the following construction satisfies the given requirements:
For each $n \in \mathbb{P}$, either $n \in A_q$ for some $q \in \{0, 1, 2, \cdots, p-1\}$, or $n \in B$.
If $n \in A_q$, we define: $L_n = \{r_0(n), r_1(n+q), r_2(n+2q), \cdots, r_k(n+kq), \cdots, r_{p-1}(n+(p-1)q), p^2 + q\}$
Else if $n \in B$, then $n = p^2 + r$ for some $r \in \{0, 1, 2, \cdots, p\}$. Then we define: $L_n = A_r \cup \{p^2 + p\}$.

For an illustration of this construction, refer to the example diagram below:

				Points																														
				A0					A1					A2					A3					A4					B					
	Q	R		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Lines	0	0	0	0					5					10					15					20					25					
	0	1	1		1					6					11					16					21				25					
	0	2	2			2					7					12					17					22			25					
	0	3	3				3					8					13					18					23		25					
	0	4	4					4					9					14					19					24	25					
	1	0	5	0						6						12						18						24		26				
	1	1	6		1						7						13						19	20						26				
	1	2	7			2						8						14	15						21					26				
	1	3	8				3						9	10						16						22				26				
	1	4	9					4	5						11						17						23			26				
	2	0	10	0							7							14		16							23				27			
	2	1	11		1							8		10							17							24			27			
	2	2	12			2							9		11							18		20							27			
	2	3	13				3		5							12							19		21						27			
	2	4	14					4		6							13		15							22					27			
	3	0	15	0								8			11								19			22						28		
	3	1	16		1								9			12			15								23					28		
	3	2	17			2			5								13			16								24				28		
	3	3	18				3			6								14			17			20								28		
	3	4	19					4			7			10								18			21							28		
	4	0	20	0									9				13				17				21								29	
	4	1	21		1				5									14				18				22							29	
	4	2	22			2				6				10									19				23						29	
	4	3	23				3				7				11				15									24					29	
	4	4	24					4				8				12				16				20									29	
			25	0	1	2	3	4																										30
			26						5	6	7	8	9																					30
			27											10	11	12	13	14																30
			28																15	16	17	18	19											30
			29																					20	21	22	23	24						30
			30																										25	26	27	28	29	30