Notation: For any $m, n \in \mathbb{R}$ such that $m^2 - 4n < 0$; let $P_{[m,n]}(X) = X^2 + mX + n$.
In particular, $P_{[0,1]}(X) = X^2 + 1$.
For any $b, c \in \mathbb{R}$ such that $b^2 - 4c < 0$; let $\mathbb{C}_{[b,c]}$ be the field over $\mathbb{R}^2$ with the operations:
$(p, q) + (r, s) = (p + r, q + s)$ and $(p, q) \cdot (r, s) = (pr - c \cdot qs, ps + qr - b \cdot qs)$.
In particular, $\mathbb{C}_{[0,1]}$ is the standard construction of complex numbers.

We note that the field $\mathbb{R}[X]/(X^2 + bX + c)$ is isomorphic to $\mathbb{C}_{[b,c]}$ under the mapping $(p + qX) \mapsto (p, q)$.
Further, we can embed $\mathbb{R}$ in $\mathbb{C}_{[b,c]}$ with the mapping $t \mapsto (t, 0)$.
Similarly, we can embed $\mathbb{R}[X]$ in $\mathbb{C}_{[b,c]}[X]$ with the mapping $P(X) = \sum a_i X^i \mapsto \sum (a_i, 0) X^i$.
Henceforth we will only deal with elements in $\mathbb{C}_{[b,c]}$ and polynomials in $\mathbb{C}_{[b,c]}[x]$, respectively.
For notational convenience we may write t or $P(x)$, to denote their images under the above mappings.

Claim 1: $\mathbb{C}_{[0,1]}$ is isomorphic to $\mathbb{C}_{[b,c]}$ under the mapping $F_{[b,c]} : \mathbb{C}_{[0,1]} \to \mathbb{C}_{[b,c]}$ defined as:
$F_{[b,c]}(p, q) = (p + q\frac{b}{\sqrt{4c-b^2}}, q\frac{2}{\sqrt{4c-b^2}})$.
Proof: It suffices to check that this map preserves field operations.

Claim 2: Let $m, n \in \mathbb{R}$ such that $m^2 - 4n < 0$; and let $P_{[m,n]}(X) = X^2 + mX + n$.
For any $x \in \mathbb{C}_{[0,1]}$, $P_{[m,n]}(x) = 0$ in $\mathbb{C}_{[0,1]}$ if and only if $P_{[m,n]}(F_{[b,c]}(x)) = 0$ in $\mathbb{C}_{[b,c]}$.
Proof: It is enough to note that $F_{[b,c]}$ keeps the reals m, n unchanged, hence it commutes with $P_{[m,n]}$.

Along with F, it is useful to define another transformation:
For any $m, n \in \mathbb{R}$ such that $m^2 - 4n < 0$; we define $G_{[m,n]}(x) = -\frac{m}{2} + x\frac{\sqrt{4n-m^2}}{2}$; for any $x \in \mathbb{C}_{[b,c]}$.
Note that G is independent of the choice of domain $\mathbb{C}_{[b,c]}$.

Claim 3: For any $x \in \mathbb{C}_{[b,c]}$, $P_{[0,1]}(x) = x^2 + 1 = 0$ if and only if $P_{[m,n]}(G_{[m,n]}(x)) = 0$.
Proof: It suffices to check that $x^2 + 1 = 0$ if and only if $(-\frac{m}{2} + x\frac{\sqrt{4n-m^2}}{2})^2 + m(-\frac{m}{2} + x\frac{\sqrt{4n-m^2}}{2}) + n = 0$.

Claim 4 Let $m, n \in \mathbb{R}$ such that $m^2 - 4n < 0$. Then the the roots of $P_{[m,n]}(X) = X^2 + mX + n$ in $\mathbb{C}_{[b,c]}$ are $(u, v) = (p + q\frac{b}{\sqrt{4c-b^2}}, q\frac{2}{\sqrt{4c-b^2}})$ and $(u, v) = (p - q\frac{b}{\sqrt{4c-b^2}}, -q\frac{2}{\sqrt{4c-b^2}})$; where $p = \frac{-m}{2}$ and $q = \frac{\sqrt{4n-m^2}}{2}$.
Proof 1: We know by standard theory, that the roots of $P_{[m,n]}(x) = 0$ in $\mathbb{C}_{[0,1]}$ are $(u, v) = (\frac{-m}{2}, \pm\frac{\sqrt{4n-m^2}}{2})$.
The result follows by applying claim 2.

Proof 2: If $x = (u, v) \in \mathbb{C}_{[b,c]}$ such that $P_{[m,n]}(x) = 0$, then $x^2 + mx + n = 0$ implies:
$(u, v) \cdot (u, v) + (m, 0) \cdot (u, v) + (n, 0) = (u^2 - cv^2 + mu + n, 2uv - bv^2 + mv) = (0, 0)$.
Noting that v cannot be zero, from the second component we get $u = \frac{bv-m}{2}$.
On substitution into the first component, and simplification, we get $v^2(b^2 - 4c) = m^2 - 4n$.
This directly leads to the two solutions given in the claim.

Claim 2 allows us to change the domain while keeping the equation the same; whereas
Claim 3 allows us to change the equation while keeping the domain the same.
Claim 4 directly gives us the roots of any equation in any domain.
Along with the known fact that $P_{[0,1]}(x) = x^2 + 1 = 0$ has the roots $(0, 1), (0, -1)$ in $\mathbb{C}_{[0,1]}$;
we can write the roots of $P_{[0,1]}(x)$, $P_{[b,c]}(x)$ and $P_{[m,n]}(x)$ in each of the domains $\mathbb{C}_{[0,1]}$ and $\mathbb{C}_{[b,c]}$:

Equation	Roots in $\mathbb{C}_{[0,1]}$	Roots in $\mathbb{C}_{[b,c]}$
$P_{[0,1]}(x) = x^2 + 1 = 0$	$(0, 1), (0, -1)$	$(\frac{b}{\sqrt{4c-b^2}}, \frac{2}{\sqrt{4c-b^2}}), (\frac{-b}{\sqrt{4c-b^2}}, \frac{-2}{\sqrt{4c-b^2}})$
$P_{[b,c]}(x) = x^2 + bx + c = 0$	$(\frac{-b}{2}, \frac{\sqrt{4c-b^2}}{2}), (\frac{-b}{2}, \frac{-\sqrt{4c-b^2}}{2})$	$(0, 1), (-b, -1)$
$P_{[m,n]}(x) = x^2 + mx + n = 0$	$(\frac{-m}{2}, \frac{\sqrt{4n-m^2}}{2}), (\frac{-m}{2}, \frac{-\sqrt{4n-m^2}}{2})$	$(\frac{-m}{2} + \frac{b\sqrt{4n-m^2}}{2\sqrt{4c-b^2}}, \frac{\sqrt{4n-m^2}}{\sqrt{4c-b^2}}), (\frac{-m}{2}, -\frac{\sqrt{4n-m^2}}{2})$

The columns for $\mathbb{C}_{[0,1]}$ and $\mathbb{C}_{[b,c]}$ are related by $F_{[b,c]}(p,q) = (p + q\frac{b}{\sqrt{4c-b^2}}, q\frac{2}{\sqrt{4c-b^2}})$
and its inverse map $F^{-1}_{[b,c]}(p,q) = (p - q\frac{b}{2}, q\frac{\sqrt{4c-b^2}}{2})$.
Similarly, the rows for $P_{[0,1]}(x)$ and $P_{[b,c]}(x)$ are related by $G_{[b,c]}(p,q) = (-\frac{b}{2} + p\frac{\sqrt{4c-b^2}}{2}, q\frac{\sqrt{4c-b^2}}{2})$
and its inverse map $G^{-1}_{[b,c]}(p,q) = (\frac{b}{\sqrt{4c-b^2}} + p\frac{2}{\sqrt{4c-b^2}}, q\frac{2}{\sqrt{4c-b^2}})$.

Further, if we denote each $(p,q) \in \mathbb{R}^2$ by the matrix $\left[\begin{smallmatrix} 1 & p \\ 0 & q \end{smallmatrix}\right]$, then $F_{[b,c]}$ can be interpreted as left-multiplication by $\mathcal{F} = \left[\begin{smallmatrix} 1 & \frac{b}{\sqrt{4c-b^2}} \\ 0 & \frac{2}{\sqrt{4c-b^2}} \end{smallmatrix}\right]$; whereas $G_{[b,c]}$ can be interpreted as right-multiplication by $\mathcal{G} = \left[\begin{smallmatrix} 1 & -\frac{b}{2} \\ 0 & \frac{\sqrt{4c-b^2}}{2} \end{smallmatrix}\right]$.

Now, we observe that $\mathcal{F} = \mathcal{G}^{-1}$; hence if $\mathcal{I} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\mathcal{J} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, then we can rewrite the above table in matrix notation as:

Equation	Roots in $\mathbb{C}_{[0,1]}$	Roots in $\mathbb{C}_{[b,c]}$
$P_{[0,1]}(x) = x^2 + 1 = 0$	$\mathcal{I}, \mathcal{J}$	$\mathcal{G}^{-1}\mathcal{I}, \mathcal{G}^{-1}\mathcal{J}$
$P_{[b,c]}(x) = x^2 + bx + c = 0$	$\mathcal{I}\mathcal{G}, \mathcal{J}\mathcal{G}$	$\mathcal{G}^{-1}\mathcal{I}\mathcal{G}, \mathcal{G}^{-1}\mathcal{J}\mathcal{G}$

Consequently, if we now multiply the matrix for a root of $P_{[0,1]}$ in $\mathbb{C}_{[b,c]}$, with the matrix for the corresponding root of $P_{[b,c]}$ in $\mathbb{C}_{[0,1]}$; we get $\mathcal{G}^{-1}\mathcal{I} \cdot \mathcal{I}\mathcal{G} = \mathcal{I}$; and similarly $\mathcal{G}^{-1}\mathcal{J} \cdot \mathcal{J}\mathcal{G} = \mathcal{I}$.